

Counting, The Caveperson Method extra

Clear, simple, and **useless** in every day use: $0, S(0), S(S(0)), S(S(S(0))), \dots$

Using Peano axioms we can prove things like

- S(S(0)) + S(S(S(0))) = S(S(S(S(S(0)))))
- $S(S(0)) \times S(S(S(S(0))) = S(S(S(S(S(S(0))))))$

Everything taught at school can be proved

And a useful theorem

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Improving on the Caveperson Method extra

- Several notations for numbers were devised over time
- Humanity settled on the (decimal) positional notation
- The **decimal** is the less important fact
- The **position** idea is very important
- Mere humans can (could?) **compute** using:
 - ► Addition table
 - Multiplication table

Addition in different methods

Roman	Decimal
V + V = X	5 + 5 = 10
XV + XV = XXX	15 + 15 = 30
XXV + XV = XL	25 + 15 = 40

The default radix is S(S(S(S(S(S(S(S(S(S(S(0)))))))))).

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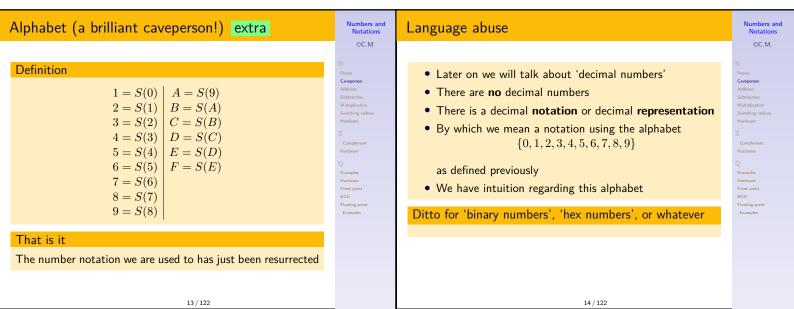
Numbers and (Failing to explain) Our Decimal Notation extra Division with Remainder extra ©C.M. ©C.M. We take for granted the alphabet 0, 1, 2, 3, 4, 5, 6, 7, 8, 9Theorem Let x and b be natural numbers such that b > S(0). Then Positioning method there are unique natural numbers q < x and d < b such that Example: 235 $x = q \times b + d$. ullet We tend to say that 235 is a natural number Corollary • However, 235 is definitely not a number Let x and b be natural numbers such that b > S(0). Then • It is a concatenation of the symbols '2', '3' and '5' there is a unique tuple $\langle d_{n-1},\dots,d_0 \rangle$, where $d_i < b$ for each i < n, such that $x = d_{n-1} \times b^{n-1} + \dots + d_1 \times b^1 + d_0 \times b^0$. • We explain: We **mean** 200 + 30 + 5• Same problem. 200, 30 and 5 are strings Definition • We mean $2 \times 10^2 + 3 \times 10 + 5$ $\langle d_{n-1},\ldots,d_0 \rangle$ is the b-radix notation for x and we write • Same problem. 10 is a string. $x = (d_{n-1} \cdots d_0)_b.$ Luckily we have the useless notation!

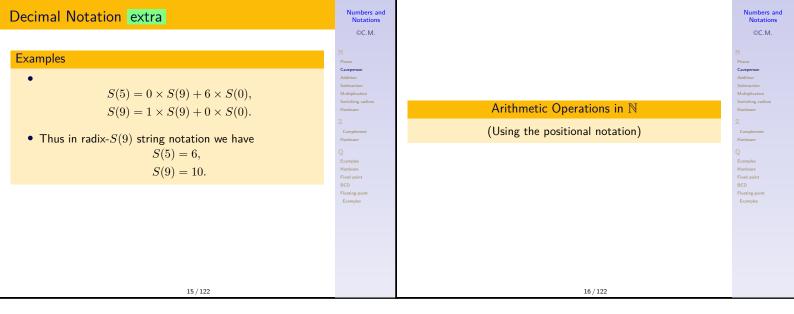
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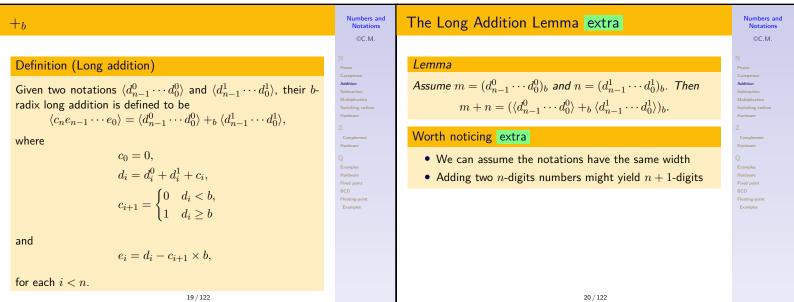
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Numbers and Notations ○ The operations + and × have been defined ○ Positional notation has been defined ○ We define + b and × b between notations ○ Then we identify + with + b and × with × b ○ Campine Hardinger Figure Fig

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Proof of the Long Addition Lemma (1) extra

Begin with the obvious:

$$\sum_{i=0}^{n-1} d_i^0 \times b^i + \sum_{i=0}^{n-1} d_i^1 \times b^i = \sum_{i=0}^{n-1} (d_i^0 + d_i^1) \times b^i$$

This does not yield a b-radix notation since we might have i's for which $d_i^0 + d_i^1 \ge b$.

We use the $\stackrel{\circ}{e_i}$ and $\stackrel{\circ}{c_i}$ notation from the definition of $+_b$ and prove by induction that for each $k \leq n$,

$$m + n = \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 + d_i^1) \times b^i.$$

The point is that $e_i < b$ and $c_i < 2$, thus when k = n the theorem is proved.

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Proof of the Long Addition Lemma (2) extra

The case k = 0 is degenerate. Assuming k < n we prove the

$$\sum_{i=0}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + (d_k^0 + d_k^1) \times b^k -$$

$$c_{k+1} \times b^{k+1} + c_{k+1} \times b^{k+1} +$$

$$\sum_{i=0}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

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Proof of the Long Addition Lemma (3) extra

$$\begin{split} = & \sum_{i=0}^{k-1} e_i \times b^i + (d_k^0 + d_k^1 + c_k - c_{k+1} \times b) \times b^k \\ & + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i = \\ = & \sum_{i=0}^{k-1} e_i \times b^i + e_k \times b^k + c_{k+1} \times b^{k+1} + \\ & \sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i = \\ = & \sum_{i=0}^{k} e_i \times b^i + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i. \quad \Box \end{split}$$

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The Long Addition Lemma, Meaning

Addition table for digits is enough

Addition of large numbers, no matter how large, can be calculated using a small addition table

This was a serious breakthrough!

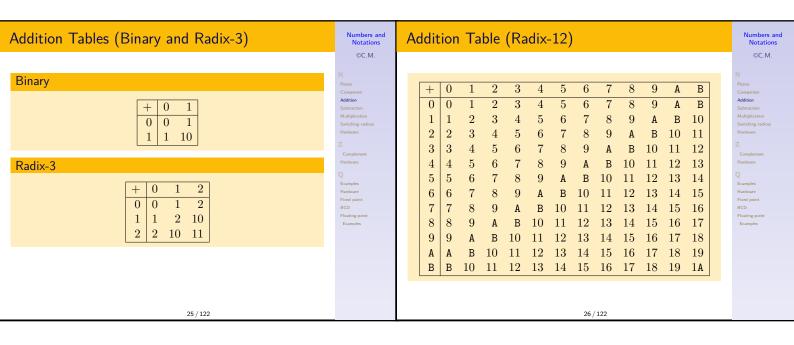
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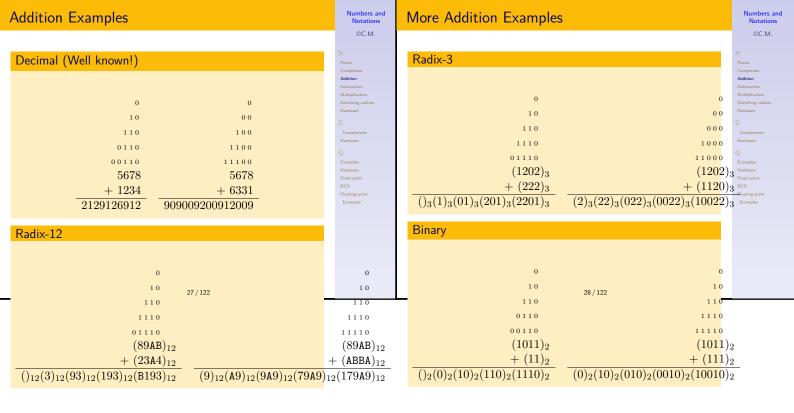
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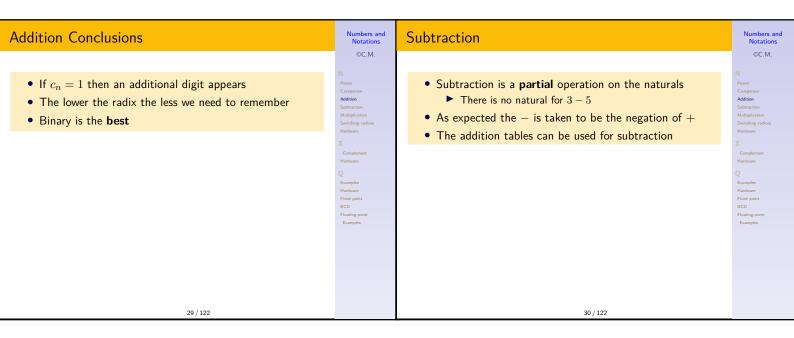
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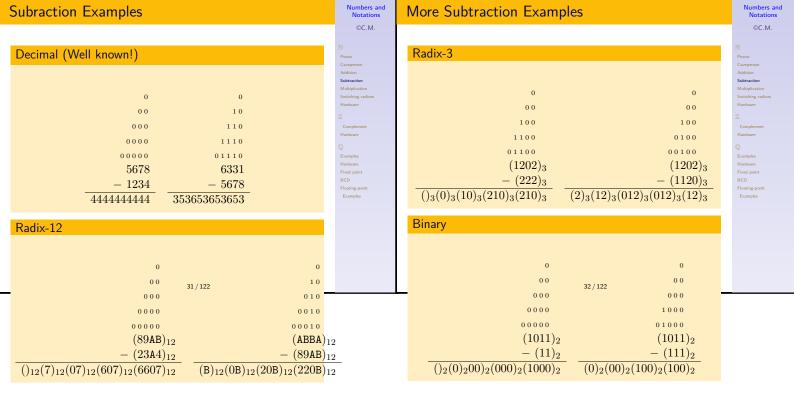
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Semi \times_b extra

Definition (Semi long multiplication)

$$\langle d_{n-1}^0 \cdots d_0^0 \rangle \times_b \langle d \rangle = \langle c_n e_{n-1} \cdots e_0 \rangle,$$

where

$$c_0 = 0,$$

$$d_i = d_i^0 \times d + c_i,$$

$$c_{i+1} = \lfloor d_i/b \rfloor$$

and

$$e_i = d_i - c_{i+1} \times b,$$

for each i < n.

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The Semi Long Multiplication Lemma extra

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Lemma

Assume
$$x = (d_{n-1}^0 \cdots d_0^0)_b$$
. Then

$$x \times (d)_b = (\langle d_{n-1}^0 \cdots d_0^0 \rangle \times_b \langle d \rangle)_b.$$

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Proof of the 1st Multiplication Lemma (1) extra

Begin with the obvious:

$$x \times (d)_b = \left(\sum_{i=0}^{n-1} d_i^0 \times b^i\right) \times d =$$
$$= \sum_{i=0}^{n-1} (d_i^0 \times d) \times b^i.$$

This does not yield a b-radix notation since we might have i's for which $d_i^0 \times d \ge b$.

Using the c_i and e_i from the definition of \times_b we will prove by induction that for each $k \leq n$,

$$m + n = \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 \times d) \times b^i.$$

The point is that $e_i, c_i < b$, thus when k = n the theorem is proved. 35 / 122

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Proof of the 1st Multi. Lemma (2) extra

The case k = 0 is degenerate. Assuming k < n we prove the case k+1.

$$\sum_{i=0}^{n-1} (d_i^0 \times d) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 \times d) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + (d_k^0 \times d) \times b^k -$$

$$c_{k+1} \times b^{k+1} + c_{k+1} \times b^{k+1} +$$

$$\sum_{i=0}^{n-1} (d_i^0 \times d) \times b^i =$$

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Proof of the 1st Multi. Lemma (3) extra

$$\begin{split} = & \sum_{i=0}^{k-1} e_i \times b^i + (d_k^0 \times d + c_k - c_{k+1} \times b) \times b^k \\ & + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i = \\ = & \sum_{i=0}^{k-1} e_i \times b^i + e_k \times b^k + c_{k+1} \times b^{k+1} + \\ & \sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i = \\ = & \sum_{i=0}^{k} e_i \times b^i + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i. \quad \Box \end{split}$$

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The 'Meaning' of the 1st Multi. Lemma extra

Knowing the multiplication table for digits is enough

Multiplication of a number, no matter the number digits in its representation, by a digit, can be calculated using the 1st multiplication lemma

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The Multiplication Lemma extra

Lemma

Fix $x=(d^0_{n_0-1}\cdots d^0_0)_b$ and $y=(d^1_{n_1-1}\cdots d^1_0)_b$. Then there is c^j and $e^j_0,\ldots,e^j_{n_0-1}$ for each $j< n_1$ such that

$$x \times y = \sum_{j=0}^{n_1-1} (c^j e_{n_0-1}^j \cdots e_0^j)_b \times b^j.$$

- ullet Note that multiplying by b^j means adding j zeros to the right of the representation
- Thus long multiplication is reduced to long addition, for which we have a lemma

Proof of the Multiplication Lemma (1) extra

 $x \times y = (d_{n_0-1}^0 \cdots d_0^0)_b \times (d_{n_1-1}^1 \cdots d_0^1)_b =$ $= \sum_{i=0}^{n_1-1} (d_{n_0-1}^0 \cdots d_0^0)_b \times d_i^1 \times b^i.$

By the 1st multiplication lemma there is $c^j, e^j_{n_0-1}, \dots, e^j_0$ such

$$(d_{n_0-1}^0 \cdots d_0^0)_b \times d_j^1 = (c^j e_{n_0-1}^j \cdots e_0^j)_b.$$

Thus

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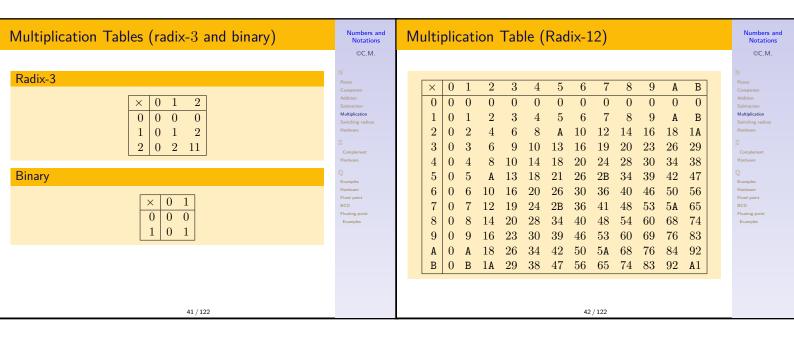
$$x \times y = \sum_{j=0}^{n_1-1} (c^j e_{n_0-1}^j \cdots e_0^j)_b \times b^j.$$

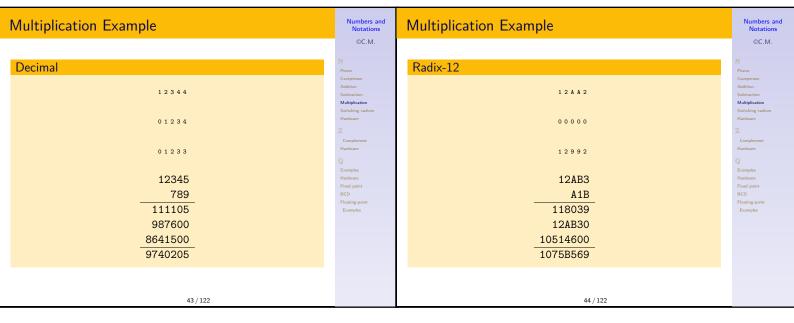
We are done.

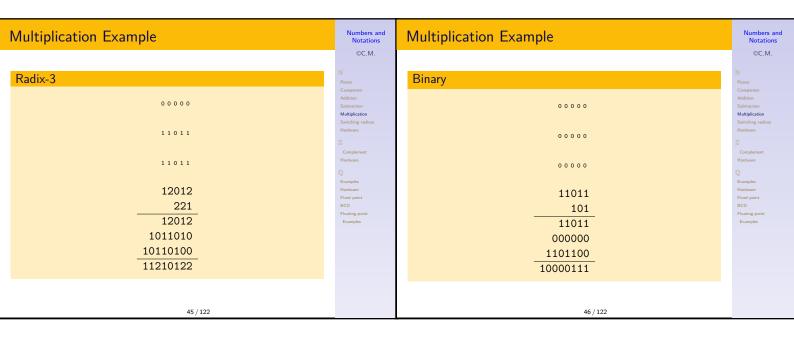
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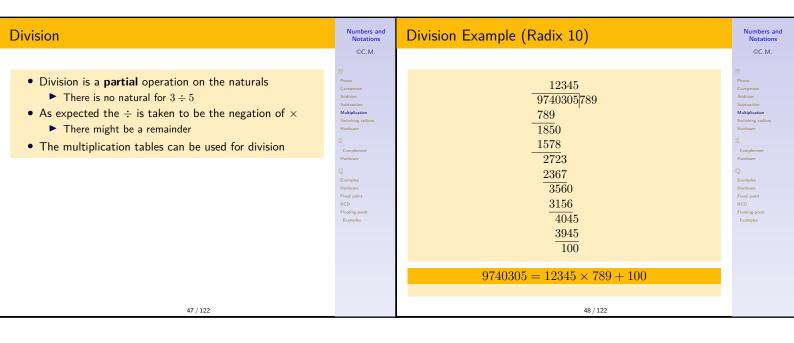
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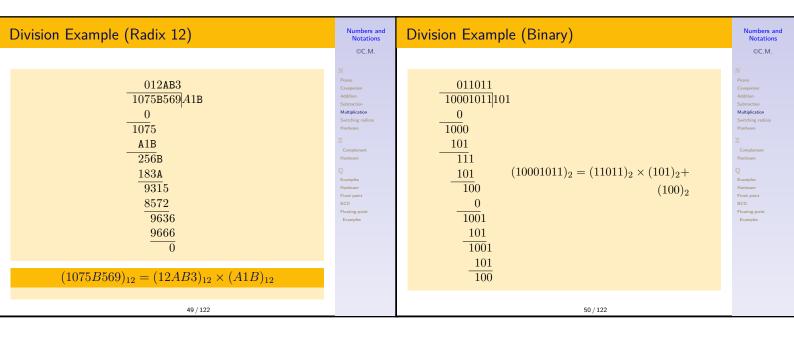
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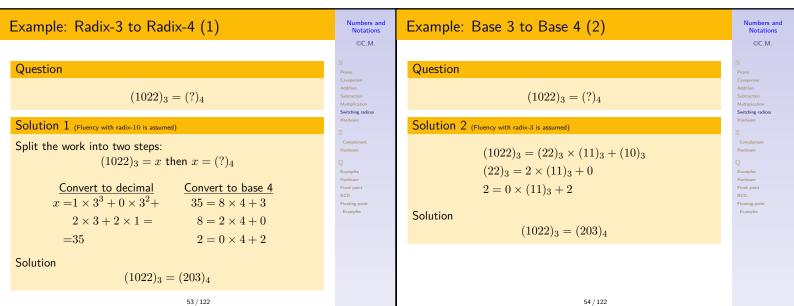


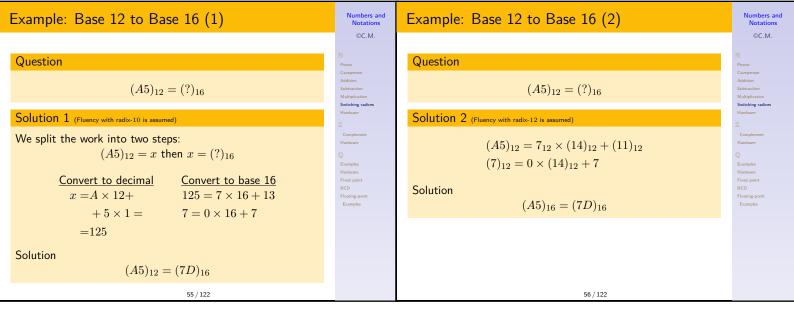


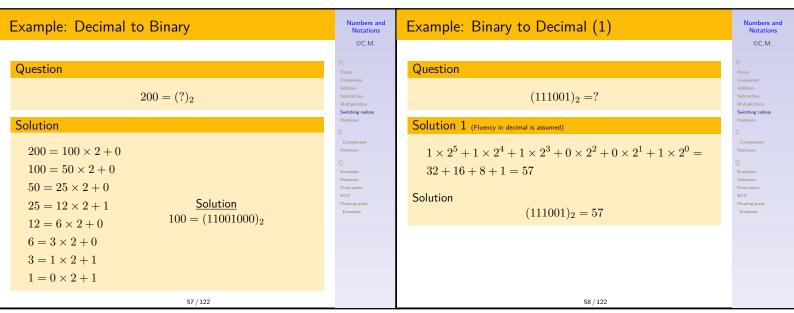


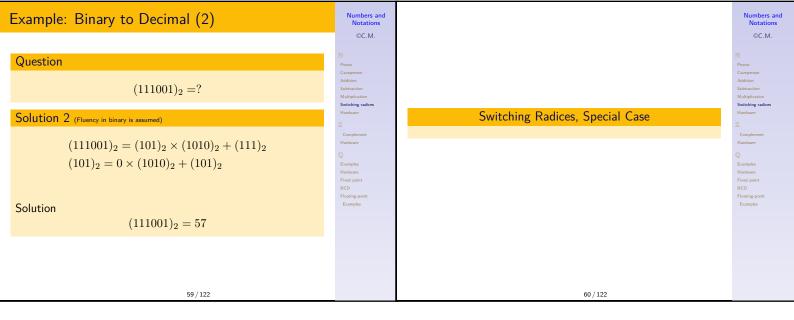




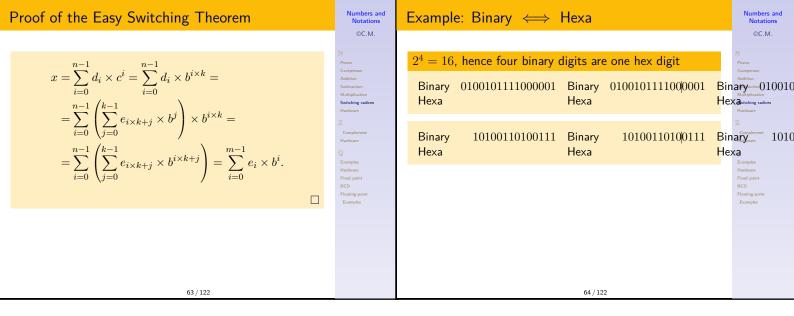


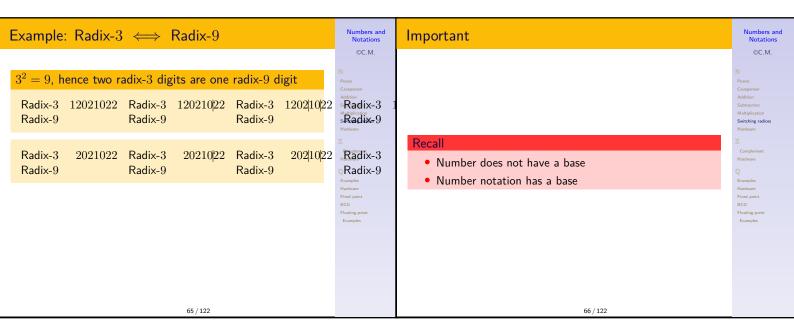


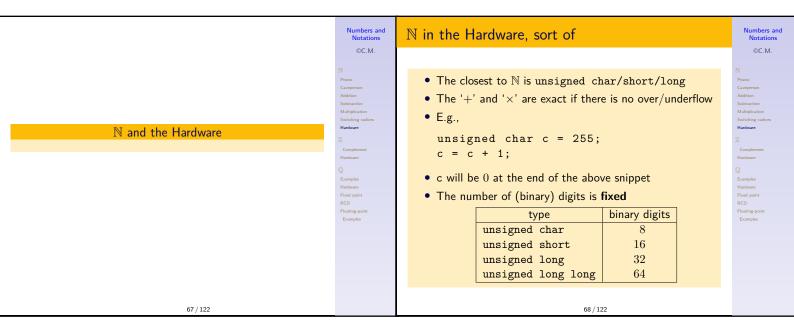


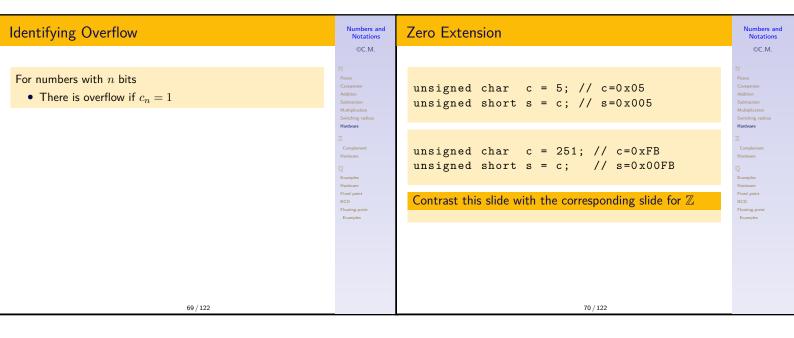


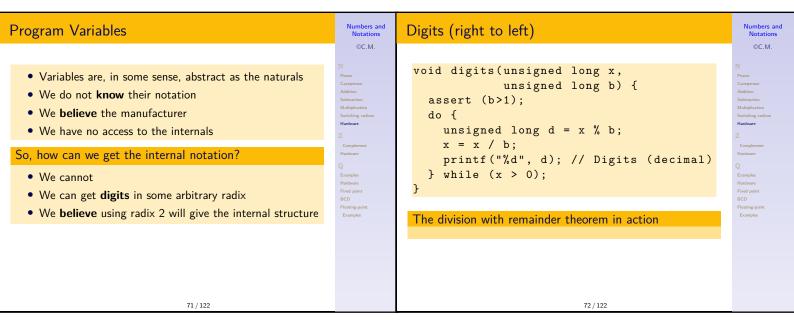
Hard Work The Easy Switching Theorem ©C.M. Lemma • It is rather clear that switching radix requires work Peano Caveperson Addition Addition Multiplication Switching rad Hardware Complement Hardware Q Examples Hardware Fixed point BCD Floating-point Examples Assume $c=b^k$, $b,k\in\mathbb{N}$, $k\geq 1$ and b>1. Let • When working in binary, we have long numbers • Converting these numers to decimal and back is not $x = (d_{n-1} \cdots d_0)_c.$ friendly Then $x = (e_{m-1} \cdots e_0)_b,$ where $m = n \times k$ and for each i < n, $d_i = (e_{i \times k + k - 1} \cdots e_{i \times k + 1} e_{i \times k})_b.$ 61 / 122 62 / 122

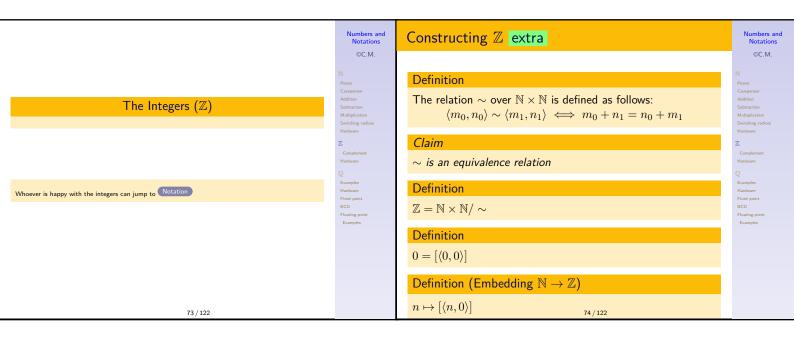


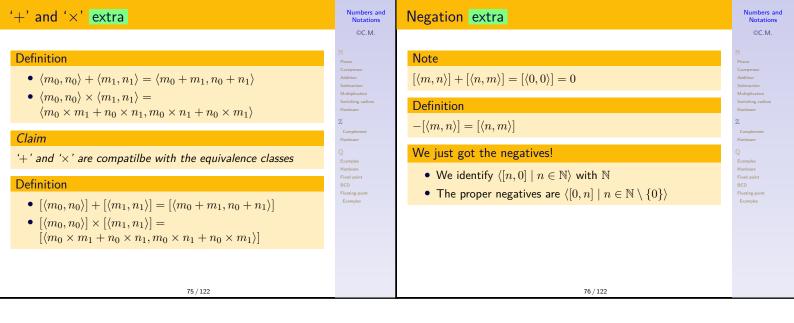


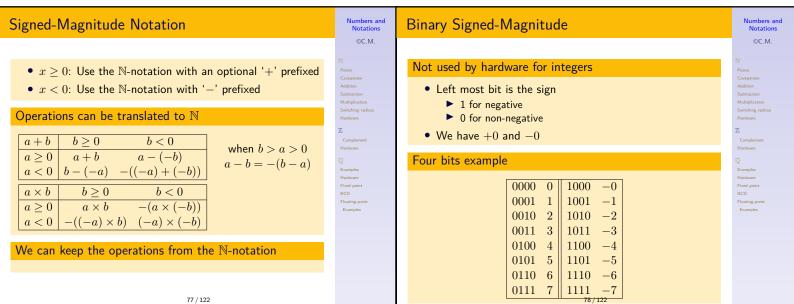


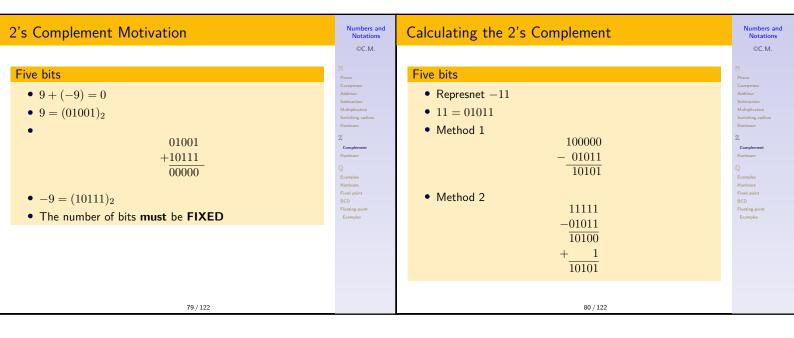




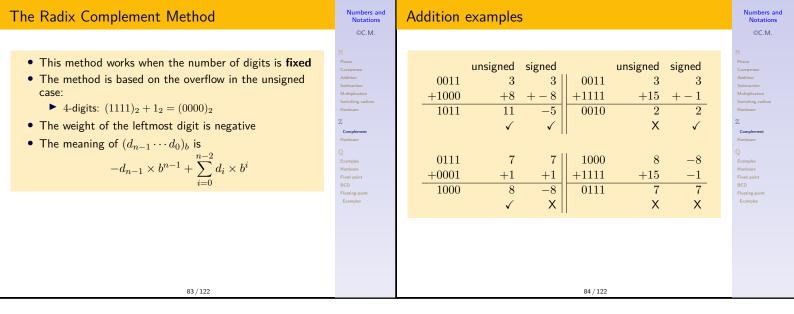


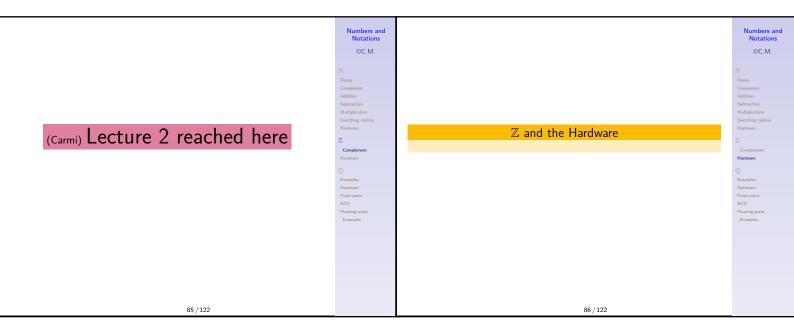


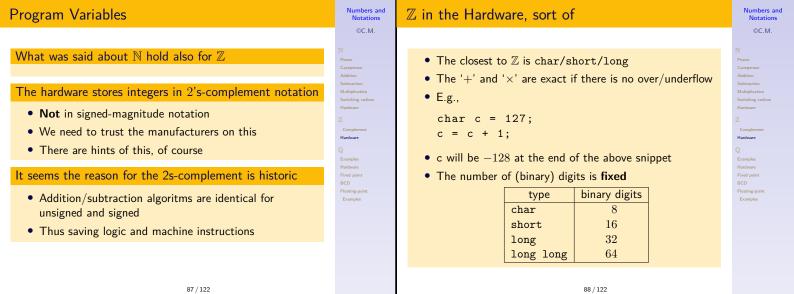




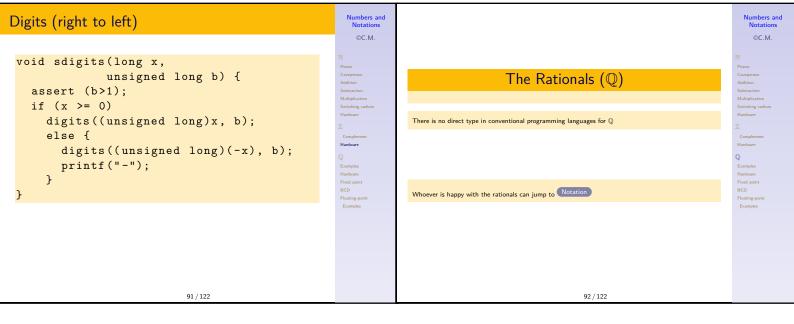
2's Complement to Decimal Table ©C.M. ©C.M. Five bits Four bits example • 00110 0000 1000 0 ► Left most bit is zero. Usual binary: 6 00011001 -7• 11001 0010 2 1010 -6▶ Left most bit is one. Calculate 2's complement: 0011 3 1011 -50100 - 41100 -4100000 -__11001 -30101 5 1101 00110 0110 6 1110 -20111 7 1111 Hence: -6 • Note the pathology at 10000 81 / 122 82 / 122







Identifying Overflow Sign Extension ©C.M. ©C.M. Assume n is the number of binary digits and c_i are the carry Peano Caveperson Caveperson Addition Subtraction Multiplication Subtraction Multiplication Subtraction Multiplication Examples Hardware Q Examples Hardware Fixed point BCD Filoating-point Examples // c = 0xFBchar c = -5;short s = с; // s = 0xFFFB• Unsigned overflow: $c_n = 1$ • Sign overflow: $c_n \neq c_{n-1}$ c = 5;// c = 0x05char • All hardware we know of ignore overflow short s = c; // s = 0x0005• There is hardware exposing the information to the interested (e.g., x86) • (No one ever look there) • There is hardware which do not (e.g., riscv) 89 / 122 90 / 122



Definition (\sim)

The relation \sim over $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is defined by $\langle m_0, n_0 \rangle \sim \langle m_1, n_1 \rangle \iff m_0 \times n_1 = n_0 \times m_1$

Definition

$$\mathbb{Q} = \left(\mathbb{Z} \times \left(\mathbb{Z} \setminus \{0\}\right)\right)/\sim$$

Definition

$$\frac{m}{n} = [\langle m, n \rangle]$$

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obers and Operations extra

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Definition

$$\frac{m_0}{n_0} + \frac{m_1}{n_1} = \left[\frac{m_0 \times n_1 + m_1 \times n_0}{n_0 \times n_1} \right]$$

Definition

$$\frac{m_0}{n_0} \times \frac{m_1}{n_1} = \left[\frac{m_0 \times m_1}{n_0 \times n_1}\right]$$

Definition (Embedding $\mathbb{Z} \to \mathbb{Q}$)

$$n\mapsto \frac{n}{1}$$

Lots of things to prove above!

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Wish we had Division with Remainder extra

Lemma

Assume $0 \leq x < 1$, $b \in \mathbb{N}$ and b > 1. Then there is unique $d \in \mathbb{N}$ and f < 1, where d < b, such that $x \times b = d + f$.

Corollary

Assume $0 \le x < 1$, $b \in \mathbb{N}$ and b > 1. Then there is a sequence $\{d_n\}_{n=-1}^{-\infty}$ such that

$$x = \sum_{n=-1}^{-\infty} d_n \times b^n.$$

 $0.\{d_n\}_{n=-1}^{-\infty}$ is the b-radix notation of x

It is not **really** infinite extra

Definition

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A notation $0.(d_n)_{n=-1}^{-\infty}$ is said to be repetitive if there is $k,l\in\mathbb{N}$ such that for each $i\in\mathbb{N}$ and j< l, $d_{k+i\times l+j}=d_{k+j}$

Lemma

The b-radix notation of $x \in \mathbb{Q}$, $0 \le x < 1$ is repetitive.

Proof

For each d_n there is a corresponding f_n . For each f_n there is a corresponding r_n such that $f_n=\frac{r_n}{m}$. Thus after at most m steps there is k,n such that $r_k=r_n$, hence $f_k=f_n$ and we find the repetition. \square

Notation

$$0.d_1 \cdots d_k \overline{d_{k+1} \cdots d_n} = 0.d_1 \cdots d_k d_{k+1} \cdots d_n d_{k+1} \cdots d_n \cdots$$

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Example: Fraction to Decimal

Question

$$\frac{7}{25} = (?)_{10}$$

Solution

$$\frac{7}{25} \times 10 = 2 + \frac{20}{25}$$
$$\frac{20}{25} \times 10 = 8 + 0$$

Thus $\frac{7}{25}=0.28.$

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Example: Radix-5 to Decimal (1)

Question

$$(0.34)_5 = (?)_{10}$$

Solution 1

$$(0.34)_5 = \frac{3}{5} + \frac{4}{25} = \frac{19}{25}$$

then

$$\frac{19}{25} \times 10 = 7 + \frac{15}{25}$$
$$\frac{15}{25} \times 10 = 6 + 0$$

Thus $(0.34)_5 = 0.76$.

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Example: Radix-5 fraction to Decimal (2)

Question

$$(0.34)_5 = (?)_{10}$$

Solution 2

$$(0.34)_5 \times (20)_5 = (12)_5 + (0.3)_5$$

 $(0.3)_5 \times (20)_5 = (11)_5 + 0$

Thus $(0.34)_5 = 0.76$

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Example: Decimal to Radix-5

Question

$$0.5 = (?)_5$$

Solution

 $0.5 \times 5 = 2 + 0.5$ // We are in infinite loop

Thus $0.5 = (0.222...)_5 = (0.\overline{2})_5$.

Imprecision is almost inevitable

since we do not possess infinite memories

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Example: Fraction to Binary

Question

$$\frac{1}{5} = (?)_2$$

Solution

$$\begin{aligned} \frac{1}{5}\times2&=0+\frac{2}{5}\\ \frac{2}{5}\times2&=0+\frac{4}{5}\\ \frac{4}{5}\times2&=1+\frac{3}{5}\\ \frac{3}{5}\times2&=1+\frac{1}{5} \text{ here comes the inifite loop} \end{aligned}$$
 Thus $\frac{1}{5}=(0.\overline{0011})_2.$

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Hardware
Fixed point
BCD
Floating-point
Examples

Example: Fraction to Binary

Question

$$\frac{19}{28} = (?)_2$$

Solution

$$\frac{19}{28} \times 2 = 1 + \frac{10}{28} \quad \frac{20}{28} \times 2 = 1 + \frac{12}{28}$$

$$\frac{10}{28} \times 2 = 0 + \frac{20}{28} \quad \frac{12}{28} \times 2 = 0 + \frac{24}{28} \quad \frac{19}{28} = (0.10\overline{101})_2$$

$$\frac{24}{28} \times 2 = 1 + \frac{20}{28}$$

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Numbers and Notations

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Peano
Caveperson
Addition
Subtraction
Multiplication
Switching radices
Hardware

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Operations

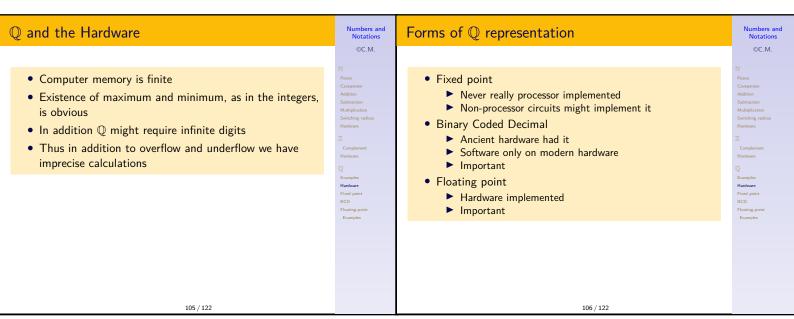
- \bullet The operations can be reduced to $\mathbb{Z}\text{-}\mathsf{operations}$
- Thus $\mathbb Z$ notation claims can be used.

$$(0.3)_4 + (1.25)_4 = \frac{(30)_4 + (125)_4}{(100)_4}$$

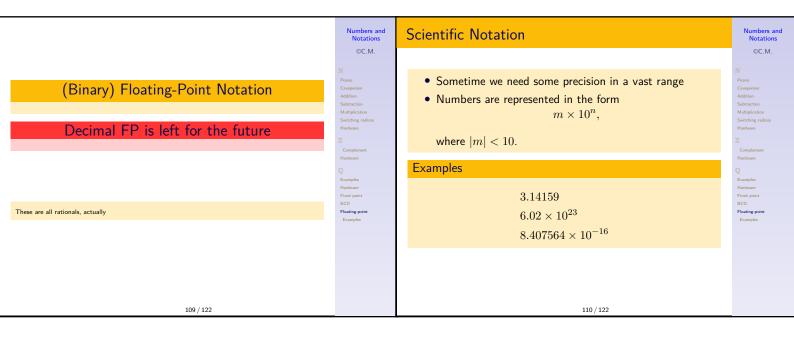
$$(0.3)_4 \times (1.25)_4 = \frac{(3)_4 \times (125)_4}{(1000)_4}$$

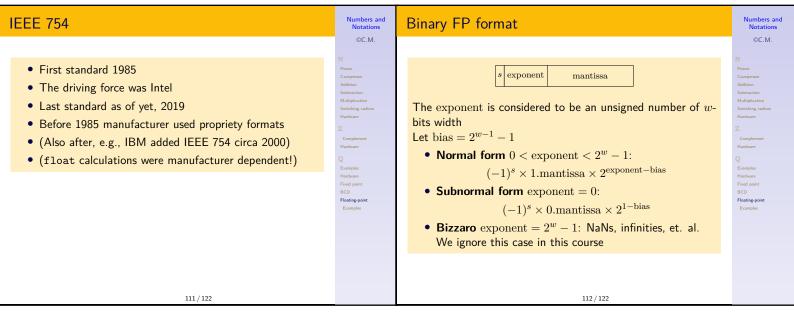
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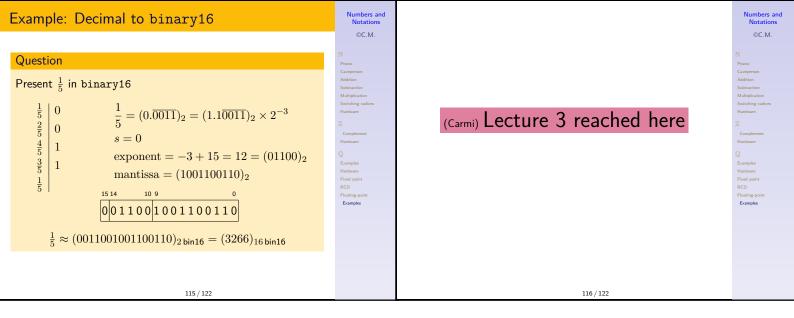


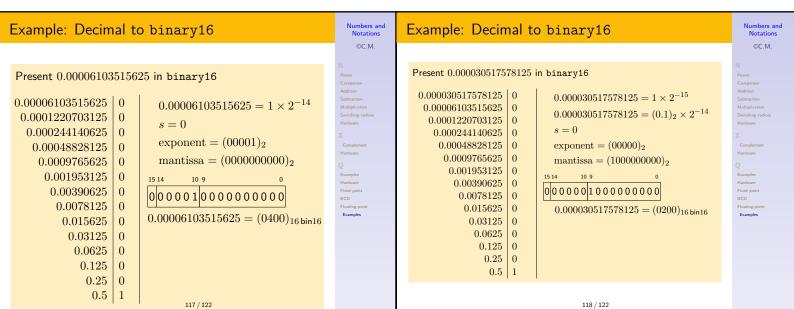


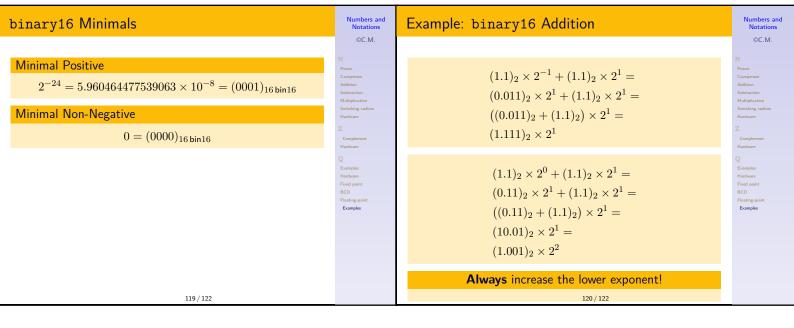


Binary FP fields width Example: Decimal to binary16 ©C.M. ©C.M. Question width of Present 200 in binary16 exponent mantissa 1 5 10 binary16 8 23 200 $200 = (11001000)_2$ binary32 1 float 100 0 52 11 double binary64 1 $(11001000)_2 = (1.1001000)_2 \times 2^7$ 50 0 15 112 binary128 1 s = 025 1 237 binary256 18 12 0 $\exp = 7 + 15 = 22 = (10110)_2$ 6 0 $mantissa = (1001000000)_2$ 3 1 1 1 0 1 0 1 1 0 1 0 0 1 0 0 0 0 0 0 0 $200 = (0101101001000000)_{2\,\mathrm{bin16}} = (5A40)_{16\,\mathrm{bin16}}$

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