Numbers and Notations

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Motivation

- We all believe we know what a number is
- Programs are full of numbers
- At the end, a program is a list of numbers
- We are being told computers use the binary method
- There are even hints of this (<<, >>, &, |, etc...)
- So we need to see how (if!) this is done

The Natural Numbers ($\mathbb{N}$)

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk

[Natural numbers were created by God, everything else is the work of men]
(attributed to) Kronecker

Whoever is happy with the naturals can skip to notation

The Naturals

- Nothing is **natural** with the natural numbers
- Intuition has been distilled in us from youth
- Intuition is useful, but it might lead to mistakes
- Formalization of the intuition saves problems
  - extra

Peano Axioms extra

- 0 is a constant symbol and S is a unary operator
- 1. $S(n) = S(m) \implies n = m$
  2. For each n, $S(n) \neq 0$
  3. For a set P, if
     - $0 \in P$ and
     - for each n, $n \in P \implies S(n) \in P$

  then P is the set of all the naturals

Intuition

- 0 is the symbol for the minimal natural
- $S(n)$ is $n + 1$ (we have neither $+$ nor 1 as of yet)

$+$, $\times$, and x^y extra

Definition ($+$)

$$m + 0 = m$$
$$m + S(n) = S(m + n)$$

Definition ($\times$)

$$m \times 0 = 0$$
$$m \times S(n) = m \times n + m$$

Definition (x^y)

$$m^0 = S(0)$$
$$m^{S(n)} = m^n \times m$$

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Order extra

Definition

- $n \geq m \iff \exists k \; n = m + k$
- $n \leq m \iff m \geq n$
- $n > m \iff n \geq m$ and $n \neq m$
- $n < m \iff m > n$

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Logicians, please have mercy extra

We are well aware of the following:

- The non-categoricity when using FO induction
- The categoricity when using SO induction
- The problem with the operators definition in FO logic

Rest of humanity

All is perfectly well

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Counting, The Caveperson Method extra

Clear, simple, and **useless** in every day use:

$$0, S(0), S(S(0)), S(S(S(0))), \ldots$$

Using Peano axioms we can prove things like

- $S(S(0)) + S(S(S(0))) = S(S(S(S(S(0)))))$
- $S(S(0)) \times S(S(S(0))) = S(S(S(S(S(S(0))))))$

Everything taught at school can be proved

Improving on the Caveperson Method extra

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- Several notations for numbers were devised over time
- Humanity settled on the (decimal) positional notation
- The **decimal** is the less important fact
- The **position** idea is very important
- Mere humans can (could?) **compute** using:
  - Addition table
  - Multiplication table

Addition in different methods

Roman	Decimal
V + V = X	5 + 5 = 10
XV + XV = XXX	15 + 15 = 30
XXV + XV = XL	25 + 15 = 40

(Failing to explain) **Our** Decimal Notation extra

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We take for granted the alphabet $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

Positioning method

Example: 235

- We tend to say that 235 is a natural number
- However, 235 is definitely not a number
- It is a concatenation of the symbols '2', '3' and '5'
- We explain: We **mean** $200 + 30 + 5$
- Same problem. 200, 30 and 5 are strings
- We mean $2 \times 10^2 + 3 \times 10 + 5$
- Same problem. 10 is a string.

Luckily we have the useless notation!

And a useful theorem

Division with Remainder extra

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Theorem

Let x and b be natural numbers such that $b > S(0)$. Then there are unique natural numbers $q < x$ and $d < b$ such that

$$x = q \times b + d.$$

Corollary

Let x and b be natural numbers such that $b > S(0)$. Then there is a unique tuple $\langle d_{n-1}, \ldots, d_0 \rangle$, where $d_i < b$ for each $i < n$, such that $x = d_{n-1} \times b^{n-1} + \cdots + d_1 \times b^1 + d_0 \times b^0$.

Definition

$\langle d_{n-1}, \ldots, d_0 \rangle$ is the b-radix notation for x and we write

$$x = (d_{n-1} \cdots d_0)_b.$$

The default radix is $S(S(S(S(S(S(S(S(S(S(0))))))))))$.

Alphabet (a brilliant caveperson!) extra

Definition

$$\begin{array}{r|l} 1 = S(0) & A = S(9) \\ 2 = S(1) & B = S(A) \\ 3 = S(2) & C = S(B) \\ 4 = S(3) & D = S(C) \\ 5 = S(4) & E = S(D) \\ 6 = S(5) & F = S(E) \\ 7 = S(6) & \\ 8 = S(7) & \\ 9 = S(8) & \end{array}$$

That is it

The number notation we are used to has just been resurrected

Language abuse

- Later on we will talk about ‘decimal numbers’
- There are **no** decimal numbers
- There is a decimal **notation** or decimal **representation**
- By which we mean a notation using the alphabet
  $$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
  as defined previously
- We have intuition regarding this alphabet

Ditto for ‘binary numbers’, ‘hex numbers’, or whatever

Decimal Notation extra

Examples

- $$S(5) = 0 \times S(9) + 6 \times S(0),$$
  $$S(9) = 1 \times S(9) + 0 \times S(0).$$
- Thus in radix-$S(9)$ string notation we have
  $$S(5) = 6,$$
  $$S(9) = 10.$$

Arithmetic Operations in $\mathbb{N}$

(Using the positional notation)

The Plan extra

- The operations $+$ and $\times$ have been defined
- Positional notation has been defined
- We define $+_b$ and $\times_b$ between **notations**
- Then we identify $+$ with $+_b$ and $\times$ with $\times_b$

$+$ and $+_b$, the crux of the matter extra

- $+_b$ is how long addition is being taught at school
- $+$ is how mathematician defines addition
- Both methods leads to the same result (a theorem!)
- The $+$ definition allows us to prove properties
- With $+_b$ we really take for granted everything

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$+_b$

Definition (Long addition)

Given two notations $\langle d^0_{n-1} \cdots d^0_0 \rangle$ and $\langle d^1_{n-1} \cdots d^1_0 \rangle$, their b-radix long addition is defined to be

$$\langle c_n e_{n-1} \cdots e_0 \rangle = \langle d^0_{n-1} \cdots d^0_0 \rangle +_b \langle d^1_{n-1} \cdots d^1_0 \rangle,$$

where

$$c_0 = 0,$$
$$d_i = d^0_i + d^1_i + c_i,$$
$$c_{i+1} = \begin{cases} 0 & d_i < b, \\ 1 & d_i \geq b \end{cases}$$

and

$$e_i = d_i - c_{i+1} \times b,$$

for each $i < n$.

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The Long Addition Lemma extra

Lemma

Assume $m = (d^0_{n-1} \cdots d^0_0)_b$ *and* $n = (d^1_{n-1} \cdots d^1_0)_b$. *Then*

$$m + n = (\langle d^0_{n-1} \cdots d^0_0 \rangle +_b \langle d^1_{n-1} \cdots d^1_0 \rangle)_b.$$

Worth noticing extra

- We can assume the notations have the same width
- Adding two n-digits numbers might yield $n+1$-digits

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Proof of the Long Addition Lemma (1) extra

Begin with the obvious:

$$\sum_{i=0}^{n-1} d_i^0 \times b^i + \sum_{i=0}^{n-1} d_i^1 \times b^i = \sum_{i=0}^{n-1} (d_i^0 + d_i^1) \times b^i$$

This does not yield a b-radix notation since we might have i's for which $d_i^0 + d_i^1 \geq b$.
We use the e_i and c_i notation from the definition of $+_b$ and prove by induction that for each $k \leq n$,

$$m + n = \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 + d_i^1) \times b^i.$$

The point is that $e_i < b$ and $c_i < 2$, thus when $k = n$ the theorem is proved.

Proof of the Long Addition Lemma (2) extra

The case $k = 0$ is degenerate. Assuming $k < n$ we prove the case $k + 1$.

$$\sum_{i=0}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + (d_k^0 + d_k^1) \times b^k -$$

$$c_{k+1} \times b^{k+1} + c_{k+1} \times b^{k+1} +$$

$$\sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

Proof of the Long Addition Lemma (3) extra

$$= \sum_{i=0}^{k-1} e_i \times b^i + (d_k^0 + d_k^1 + c_k - c_{k+1} \times b) \times b^k$$

$$+ c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k-1} e_i \times b^i + e_k \times b^k + c_{k+1} \times b^{k+1} +$$

$$\sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i =$$

$$= \sum_{i=0}^{k} e_i \times b^i + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 + d_i^1) \times b^i. \quad \square$$

The Long Addition Lemma, **Meaning**

Addition table for digits is enough

Addition of large numbers, no matter how large, can be calculated using a small addition table

This was a serious breakthrough!

Addition Tables (Binary and Radix-3)

Binary

+	0	1
0	0	1
1	1	10

Radix-3

+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

Addition Table (Radix-12)

+	0	1	2	3	4	5	6	7	8	9	A	B
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A	A	B	10	11	12	13	14	15	16	17	18	19
B	B	10	11	12	13	14	15	16	17	18	19	1A

Addition Examples

Decimal (Well known!)

```
    0            0
   1 0          0 0
  1 1 0        1 0 0
 0 1 1 0      1 1 0 0
0 0 1 1 0    1 1 1 0 0
     5678          5678
   + 1234        + 6331
2129126912   909009200912009
```

Radix-12

```
    0            0
   1 0          1 0
  1 1 0        1 1 0
 1 1 1 0      1 1 1 0
0 1 1 1 0    1 1 1 1 0
```

$(89AB)_{12}$
$+ (23A4)_{12}$
$()_{12}(3)_{12}(93)_{12}(193)_{12}(B193)_{12}$

$(89AB)_{12}$
$+ (ABBA)_{12}$
$(9)_{12}(A9)_{12}(9A9)_{12}(79A9)_{12}(179A9)_{12}$

More Addition Examples

Radix-3

```
    0            0
   1 0          0 0
  1 1 0        0 0 0
 1 1 1 0      1 0 0 0
0 1 1 1 0    1 1 0 0 0
```

$(1202)_3$
$+ (222)_3$
$()_3(1)_3(01)_3(201)_3(2201)_3$

$(1202)_3$
$+ (1120)_3$
$(2)_3(22)_3(022)_3(0022)_3(10022)_3$

Binary

```
    0            0
   1 0          1 0
  1 1 0        1 1 0
 0 1 1 0      1 1 1 0
0 0 1 1 0    1 1 1 1 0
```

$(1011)_2$
$+ (11)_2$
$()_2(0)_2(10)_2(110)_2(1110)_2$

$(1011)_2$
$+ (111)_2$
$(0)_2(10)_2(010)_2(0010)_2(10010)_2$

Addition Conclusions

- If $c_n = 1$ then an additional digit appears
- The lower the radix the less we need to remember
- Binary is the **best**

Subtraction

- Subtraction is a **partial** operation on the naturals
  - There is no natural for $3 - 5$
- As expected the $-$ is taken to be the negation of $+$
- The addition tables can be used for subtraction

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Subraction Examples

Decimal (Well known!)

$$\begin{array}{r} 0 \\ 0\,0 \\ 0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 5678 \\ -\ 1234 \\ \hline 4444444444 \end{array} \qquad \begin{array}{r} 0 \\ 1\,0 \\ 1\,1\,0 \\ 1\,1\,1\,0 \\ 0\,1\,1\,1\,0 \\ 6331 \\ -\ 5678 \\ \hline 353653653653 \end{array}$$

Radix-12

$$\begin{array}{r} 0 \\ 0\,0 \\ 0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ (89\mathtt{AB})_{12} \\ -\ (23\mathtt{A}4)_{12} \\ \hline ()_{12}(7)_{12}(07)_{12}(607)_{12}(6607)_{12} \end{array} \qquad \begin{array}{r} 0 \\ 1\,0 \\ 0\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1\,0 \\ (\mathtt{ABBA})_{12} \\ -\ (89\mathtt{AB})_{12} \\ \hline (\mathtt{B})_{12}(0\mathtt{B})_{12}(20\mathtt{B})_{12}(220\mathtt{B})_{12} \end{array}$$

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More Subtraction Examples

Radix-3

$$\begin{array}{r} 0 \\ 0\,0 \\ 1\,0\,0 \\ 1\,1\,0\,0 \\ 0\,1\,1\,0\,0 \\ (1202)_3 \\ -\ (222)_3 \\ \hline ()_3(0)_3(10)_3(210)_3(210)_3 \end{array} \qquad \begin{array}{r} 0 \\ 0\,0 \\ 1\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0\,0 \\ (1202)_3 \\ -\ (1120)_3 \\ \hline (2)_3(12)_3(012)_3(012)_3(12)_3 \end{array}$$

Binary

$$\begin{array}{r} 0 \\ 0\,0 \\ 0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ (1011)_2 \\ -\ (11)_2 \\ \hline ()_2(0)_2 00)_2(000)_2(1000)_2 \end{array} \qquad \begin{array}{r} 0 \\ 0\,0 \\ 0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0\,0 \\ (1011)_2 \\ -\ (111)_2 \\ \hline (0)_2(00)_2(100)_2(100)_2 \end{array}$$

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Semi $\times_b$ extra

Definition (Semi long multiplication)

$$\langle d^0_{n-1} \cdots d^0_0 \rangle \times_b \langle d \rangle = \langle c_n e_{n-1} \cdots e_0 \rangle,$$

where

$$c_0 = 0,$$
$$d_i = d^0_i \times d + c_i,$$
$$c_{i+1} = \lfloor d_i / b \rfloor$$

and

$$e_i = d_i - c_{i+1} \times b,$$

for each $i < n$.

The Semi Long Multiplication Lemma extra

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Lemma

Assume $x = (d^0_{n-1} \cdots d^0_0)_b$. *Then*

$$x \times (d)_b = (\langle d^0_{n-1} \cdots d^0_0 \rangle \times_b \langle d \rangle)_b.$$

Proof of the 1st Multiplication Lemma (1) extra

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Begin with the obvious:

$$x \times (d)_b = \left(\sum_{i=0}^{n-1} d^0_i \times b^i \right) \times d =$$
$$= \sum_{i=0}^{n-1} (d^0_i \times d) \times b^i.$$

This does not yield a b-radix notation since we might have i's for which $d^0_i \times d \geq b$.
Using the c_i and e_i from the definition of $\times_b$ we will prove by induction that for each $k \leq n$,

$$m + n = \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d^0_i \times d) \times b^i.$$

The point is that $e_i, c_i < b$, thus when $k = n$ the theorem is proved.

Proof of the 1st Multi. Lemma (2) extra

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The case $k = 0$ is degenerate. Assuming $k < n$ we prove the case $k + 1$.

$$\sum_{i=0}^{n-1} (d^0_i \times d) \times b^i =$$
$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + \sum_{i=k}^{n-1} (d^0_i \times d) \times b^i =$$
$$= \sum_{i=0}^{k-1} e_i \times b^i + c_k \times b^k + (d^0_k \times d) \times b^k -$$
$$c_{k+1} \times b^{k+1} + c_{k+1} \times b^{k+1} +$$
$$\sum_{i=k+1}^{n-1} (d^0_i \times d) \times b^i =$$

Proof of the 1st Multi. Lemma (3) extra

$$=\sum_{i=0}^{k-1} e_i \times b^i + (d_k^0 \times d + c_k - c_{k+1} \times b) \times b^k$$
$$+c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i =$$
$$=\sum_{i=0}^{k-1} e_i \times b^i + e_k \times b^k + c_{k+1} \times b^{k+1} +$$
$$\sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i =$$
$$=\sum_{i=0}^{k} e_i \times b^i + c_{k+1} \times b^{k+1} + \sum_{i=k+1}^{n-1} (d_i^0 \times d) \times b^i. \quad \square$$

The 'Meaning' of the 1st Multi. Lemma extra

Knowing the multiplication table for digits is enough

Multiplication of a number, no matter the number digits in its representation, by a digit, can be calculated using the 1st multiplication lemma

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The Multiplication Lemma extra

Lemma

Fix $x = (d^0_{n_0-1} \cdots d^0_0)_b$ and $y = (d^1_{n_1-1} \cdots d^1_0)_b$. Then there is c^j and $e^j_0, \ldots, e^j_{n_0-1}$ for each $j < n_1$ such that

$$x \times y = \sum_{j=0}^{n_1-1} (c^j e^j_{n_0-1} \cdots e^j_0)_b \times b^j.$$

- Note that multiplying by b^j means adding j zeros to the right of the representation
- Thus long multiplication is reduced to long addition, for which we have a lemma

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Proof of the Multiplication Lemma (1) extra

$$x \times y =(d^0_{n_0-1} \cdots d^0_0)_b \times (d^1_{n_1-1} \cdots d^1_0)_b =$$
$$= \sum_{j=0}^{n_1-1} (d^0_{n_0-1} \cdots d^0_0)_b \times d^1_i \times b^i.$$

By the 1st multiplication lemma there is $c^j, e^j_{n_0-1}, \ldots, e^j_0$ such that

$$(d^0_{n_0-1} \cdots d^0_0)_b \times d^1_j = (c^j e^j_{n_0-1} \cdots e^j_0)_b.$$

Thus

$$x \times y = \sum_{j=0}^{n_1-1} (c^j e^j_{n_0-1} \cdots e^j_0)_b \times b^j.$$

We are done.

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Multiplication Tables (radix-3 and binary)

Radix-3

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

Binary

×	0	1
0	0	0
1	0	1

Multiplication Table (Radix-12)

×	0	1	2	3	4	5	6	7	8	9	A	B
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	A	B
2	0	2	4	6	8	A	10	12	14	16	18	1A
3	0	3	6	9	10	13	16	19	20	23	26	29
4	0	4	8	10	14	18	20	24	28	30	34	38
5	0	5	A	13	18	21	26	2B	34	39	42	47
6	0	6	10	16	20	26	30	36	40	46	50	56
7	0	7	12	19	24	2B	36	41	48	53	5A	65
8	0	8	14	20	28	34	40	48	54	60	68	74
9	0	9	16	23	30	39	46	53	60	69	76	83
A	0	A	18	26	34	42	50	5A	68	76	84	92
B	0	B	1A	29	38	47	56	65	74	83	92	A1

Multiplication Example

Decimal

```
  1 2 3 4 4

  0 1 2 3 4

  0 1 2 3 3

    12345
      789
  -------
   111105
   987600
  8641500
  -------
  9740205
```


Multiplication Example

Radix-12

```
  1 2 A A 2

  0 0 0 0 0

  1 2 9 9 2

     12AB3
       A1B
  --------
    118039
    12AB30
  10514600
  --------
  1075B569
```

Multiplication Example

Radix-3

```
   0 0 0 0 0
   1 1 0 1 1
   1 1 0 1 1
     12012
       221
  --------
     12012
   1011010
  10110100
  --------
  11210122
```


Multiplication Example

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Binary

```
    0 0 0 0 0
    0 0 0 0 0
    0 0 0 0 0
      11011
        101
   --------
      11011
     000000
    1101100
   --------
   10000111
```


Division

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- Division is a **partial** operation on the naturals
  - There is no natural for $3 \div 5$
- As expected the $\div$ is taken to be the negation of $\times$
  - There might be a remainder
- The multiplication tables can be used for division

Division Example (Radix 10)

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```
   12345
  -------
  9740305|789
  789
  ----
  1850
  1578
  ----
   2723
   2367
   ----
    3560
    3156
    ----
     4045
     3945
     ----
      100
```

$$9740305 = 12345 \times 789 + 100$$

Division Example (Radix 12)


```
     012AB3
  1075B569|A1B
     0
  1075
   A1B
   256B
   183A
    9315
    8572
     9636
     9666
        0
```

$(1075B569)_{12} = (12AB3)_{12} \times (A1B)_{12}$

Division Example (Binary)

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```
     011011
  10001011|101
     0
  1000
   101
    111
    101
     100
       0
     1001
      101
     1001
      101
      100
```

$(10001011)_2 = (11011)_2 \times (101)_2 + (100)_2$

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(Carmi) Lecture 1 reached here

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Switching Radices

Luckily we have the '$+$' and '$\times$' at this point

Example: Radix-3 to Radix-4 (1)

Question

$$(1022)_3 = (?)_4$$

Solution 1 (Fluency with radix-10 is assumed)

Split the work into two steps:

$$(1022)_3 = x \text{ then } x = (?)_4$$

Convert to decimal	Convert to base 4
$x = 1 \times 3^3 + 0 \times 3^2 +$	$35 = 8 \times 4 + 3$
$2 \times 3 + 2 \times 1 =$	$8 = 2 \times 4 + 0$
$= 35$	$2 = 0 \times 4 + 2$

Solution

$$(1022)_3 = (203)_4$$

Example: Base 3 to Base 4 (2)

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Question

$$(1022)_3 = (?)_4$$

Solution 2 (Fluency with radix-3 is assumed)

$$(1022)_3 = (22)_3 \times (11)_3 + (10)_3$$
$$(22)_3 = 2 \times (11)_3 + 0$$
$$2 = 0 \times (11)_3 + 2$$

Solution

$$(1022)_3 = (203)_4$$

Example: Base 12 to Base 16 (1)

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Question

$$(A5)_{12} = (?)_{16}$$

Solution 1 (Fluency with radix-10 is assumed)

We split the work into two steps:

$$(A5)_{12} = x \text{ then } x = (?)_{16}$$

Convert to decimal	Convert to base 16
$x = A \times 12 +$	$125 = 7 \times 16 + 13$
$+ 5 \times 1 =$	$7 = 0 \times 16 + 7$
$= 125$	

Solution

$$(A5)_{12} = (7D)_{16}$$

Example: Base 12 to Base 16 (2)

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Question

$$(A5)_{12} = (?)_{16}$$

Solution 2 (Fluency with radix-12 is assumed)

$$(A5)_{12} = 7_{12} \times (14)_{12} + (11)_{12}$$
$$(7)_{12} = 0 \times (14)_{12} + 7$$

Solution

$$(A5)_{16} = (7D)_{16}$$

Example: Decimal to Binary

Question

$$200 = (?)_2$$

Solution

$$200 = 100 \times 2 + 0$$
$$100 = 50 \times 2 + 0$$
$$50 = 25 \times 2 + 0$$
$$25 = 12 \times 2 + 1$$
$$12 = 6 \times 2 + 0$$
$$6 = 3 \times 2 + 0$$
$$3 = 1 \times 2 + 1$$
$$1 = 0 \times 2 + 1$$

Solution

$$100 = (11001000)_2$$

Example: Binary to Decimal (1)

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Question

$$(111001)_2 = ?$$

Solution 1 (Fluency in decimal is assumed)

$$1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 =$$
$$32 + 16 + 8 + 1 = 57$$

Solution

$$(111001)_2 = 57$$

Example: Binary to Decimal (2)

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Question

$$(111001)_2 = ?$$

Solution 2 (Fluency in binary is assumed)

$$(111001)_2 = (101)_2 \times (1010)_2 + (111)_2$$
$$(101)_2 = 0 \times (1010)_2 + (101)_2$$

Solution

$$(111001)_2 = 57$$

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Switching Radices, Special Case

Hard Work

- It is rather clear that switching radix requires work
- When working in binary, we have **long** numbers
- Converting these numers to decimal and back is not friendly

The Easy Switching Theorem

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Lemma

Assume $c = b^k$, $b, k \in \mathbb{N}$, $k \geq 1$ *and* $b > 1$. *Let*

$$x = (d_{n-1} \cdots d_0)_c.$$

Then

$$x = (e_{m-1} \cdots e_0)_b,$$

where

$$m = n \times k$$

and for each $i < n$,

$$d_i = (e_{i \times k + k - 1} \cdots e_{i \times k + 1} e_{i \times k})_b.$$

Proof of the Easy Switching Theorem

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$$x = \sum_{i=0}^{n-1} d_i \times c^i = \sum_{i=0}^{n-1} d_i \times b^{i \times k} =$$
$$= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{k-1} e_{i \times k + j} \times b^j \right) \times b^{i \times k} =$$
$$= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{k-1} e_{i \times k + j} \times b^{i \times k + j} \right) = \sum_{i=0}^{m-1} e_i \times b^i.$$

□

Example: Binary $\Longleftrightarrow$ Hexa

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$2^4 = 16$, hence four binary digits are one hex digit

Binary	0100101111000001	Binary	010010111100|0001	Binary	010010
Hexa		Hexa		Hexa	

Binary	10100110100111	Binary	1010011010|0111	Binary	1010
Hexa		Hexa		Hexa	

Example: Radix-3 $\iff$ Radix-9

$3^2 = 9$, hence two radix-3 digits are one radix-9 digit

Radix-3	12021022	Radix-3	$120210\|22$	Radix-3	$1202\|10\|22$	Radix-3	1
Radix-9		Radix-9		Radix-9		Radix-9	

Radix-3	2021022	Radix-3	$20210\|22$	Radix-3	$202\|10\|22$	Radix-3
Radix-9		Radix-9		Radix-9		Radix-9

Important

Recall

- Number does not have a base
- Number notation has a base

ℕ and the Hardware

ℕ in the Hardware, sort of

- The closest to ℕ is `unsigned char/short/long`
- The '$+$' and '$\times$' are exact if there is no over/underflow
- E.g.,

```
unsigned char c = 255;
c = c + 1;
```

- `c` will be 0 at the end of the above snippet
- The number of (binary) digits is **fixed**

type	binary digits
`unsigned char`	8
`unsigned short`	16
`unsigned long`	32
`unsigned long long`	64

Identifying Overflow

For numbers with n bits

- There is overflow if $c_n = 1$

Zero Extension

```
unsigned char  c = 5; // c=0x05
unsigned short s = c; // s=0x005
```

```
unsigned char  c = 251; // c=0xFB
unsigned short s = c;   // s=0x00FB
```

Contrast this slide with the corresponding slide for $\mathbb{Z}$

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Program Variables

- Variables are, in some sense, abstract as the naturals
- We do not **know** their notation
- We **believe** the manufacturer
- We have no access to the internals

So, how can we get the internal notation?

- We cannot
- We can get **digits** in some arbitrary radix
- We **believe** using radix 2 will give the internal structure

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Digits (right to left)

```
void digits(unsigned long x,
            unsigned long b) {
  assert (b>1);
  do {
    unsigned long d = x % b;
    x = x / b;
    printf("%d", d); // Digits (decimal)
  } while (x > 0);
}
```

The division with remainder theorem in action

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The Integers ($\mathbb{Z}$)

Whoever is happy with the integers can jump to Notation

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Constructing $\mathbb{Z}$ extra

Definition

The relation $\sim$ over $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$\langle m_0, n_0 \rangle \sim \langle m_1, n_1 \rangle \iff m_0 + n_1 = n_0 + m_1$$

Claim

$\sim$ is an equivalence relation

Definition

$\mathbb{Z} = \mathbb{N} \times \mathbb{N} / \sim$

Definition

$0 = [\langle 0, 0 \rangle]$

Definition (Embedding $\mathbb{N} \to \mathbb{Z}$)

$n \mapsto [\langle n, 0 \rangle]$

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'$+$' and '$\times$' extra

Definition

- $\langle m_0, n_0 \rangle + \langle m_1, n_1 \rangle = \langle m_0 + m_1, n_0 + n_1 \rangle$
- $\langle m_0, n_0 \rangle \times \langle m_1, n_1 \rangle =$
  $\langle m_0 \times m_1 + n_0 \times n_1, m_0 \times n_1 + n_0 \times m_1 \rangle$

Claim

'$+$' and '$\times$' are compatilbe with the equivalence classes

Definition

- $[\langle m_0, n_0 \rangle] + [\langle m_1, n_1 \rangle] = [\langle m_0 + m_1, n_0 + n_1 \rangle]$
- $[\langle m_0, n_0 \rangle] \times [\langle m_1, n_1 \rangle] =$
  $[\langle m_0 \times m_1 + n_0 \times n_1, m_0 \times n_1 + n_0 \times m_1 \rangle]$

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Negation extra

Note

$[\langle m, n \rangle] + [\langle n, m \rangle] = [\langle 0, 0 \rangle] = 0$

Definition

$-[\langle m, n \rangle] = [\langle n, m \rangle]$

We just got the negatives!

- We identify $\langle [n, 0] \mid n \in \mathbb{N} \rangle$ with $\mathbb{N}$
- The proper negatives are $\langle [0, n] \mid n \in \mathbb{N} \setminus \{0\} \rangle$

Signed-Magnitude Notation

- $x \geq 0$: Use the $\mathbb{N}$-notation with an optional '$+$' prefixed
- $x < 0$: Use the $\mathbb{N}$-notation with '$-$' prefixed

Operations can be translated to $\mathbb{N}$

$a+b$	$b \geq 0$	$b < 0$
$a \geq 0$	$a+b$	$a-(-b)$
$a < 0$	$b-(-a)$	$-((-a)+(-b))$

$a \times b$	$b \geq 0$	$b < 0$
$a \geq 0$	$a \times b$	$-(a \times (-b))$
$a < 0$	$-((-a) \times b)$	$(-a) \times (-b)$

when $b > a > 0$
$a - b = -(b - a)$

We can keep the operations from the $\mathbb{N}$-notation

Binary Signed-Magnitude

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Not used by hardware for integers

- Left most bit is the sign
  - 1 for negative
  - 0 for non-negative
- We have $+0$ and -0

Four bits example

0000	0	1000	-0
0001	1	1001	-1
0010	2	1010	-2
0011	3	1011	-3
0100	4	1100	-4
0101	5	1101	-5
0110	6	1110	-6
0111	7	1111	-7

2's Complement Motivation

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Five bits

- $9 + (-9) = 0$
- $9 = (01001)_2$
- $$\begin{array}{r} 01001 \\ +10111 \\ \hline 00000 \end{array}$$
- $-9 = (10111)_2$
- The number of bits **must** be **FIXED**

Calculating the 2's Complement

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Five bits

- Represnet -11
- $11 = 01011$
- Method 1
$$\begin{array}{r} 100000 \\ -\ \ 01011 \\ \hline 10101 \end{array}$$
- Method 2
$$\begin{array}{r} 11111 \\ -01011 \\ \hline 10100 \\ +\quad\ \ 1 \\ \hline 10101 \end{array}$$

2's Complement to Decimal

Five bits

- 00110
  - Left most bit is zero. Usual binary: 6
- 11001
  - Left most bit is one. Calculate 2's complement:

$$\begin{array}{r} 100000 \\ -\ 11001 \\ \hline 00110 \end{array}$$

  Hence: -6
- Note the pathology at 10000

Table

Four bits example

0000	0	1000	-8
0001	1	1001	-7
0010	2	1010	-6
0011	3	1011	-5
0100	4	1100	-4
0101	5	1101	-3
0110	6	1110	-2
0111	7	1111	-1

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The Radix Complement Method

- This method works when the number of digits is **fixed**
- The method is based on the overflow in the unsigned case:
  - 4-digits: $(1111)_2 + 1_2 = (0000)_2$
- The weight of the leftmost digit is negative
- The meaning of $(d_{n-1} \cdots d_0)_b$ is

$$-d_{n-1} \times b^{n-1} + \sum_{i=0}^{n-2} d_i \times b^i$$

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Addition examples

	unsigned	signed		unsigned	signed
0011	3	3	0011	3	3
+1000	+8	$+-8$	+1111	+15	$+-1$
1011	11	-5	0010	2	2
	✓	✓		X	✓
0111	7	7	1000	8	-8
+0001	+1	+1	+1111	+15	-1
1000	8	-8	0111	7	7
	✓	X		X	X

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(Carmi) Lecture 2 reached here

$\mathbb{Z}$ and the Hardware

Program Variables

What was said about $\mathbb{N}$ hold also for $\mathbb{Z}$

The hardware stores integers in 2's-complement notation

- **Not** in signed-magnitude notation
- We need to trust the manufacturers on this
- There are hints of this, of course

It seems the reason for the 2s-complement is historic

- Addition/subtraction algoritms are identical for unsigned and signed
- Thus saving logic and machine instructions

$\mathbb{Z}$ in the Hardware, sort of

- The closest to $\mathbb{Z}$ is `char/short/long`
- The '$+$' and '$\times$' are exact if there is no over/underflow
- E.g.,

```
char c = 127;
c = c + 1;
```

- c will be -128 at the end of the above snippet
- The number of (binary) digits is **fixed**

type	binary digits
`char`	8
`short`	16
`long`	32
`long long`	64

Identifying Overflow

Assume n is the number of binary digits and c_i are the carry bits

- Unsigned overflow: $c_n = 1$
- Sign overflow: $c_n \neq c_{n-1}$

- All hardware we know of ignore overflow
- There is hardware exposing the information to the interested (e.g., x86)
- (No one ever look there)
- There is hardware which do not (e.g., riscv)

Sign Extension


```
char  c = -5;  // c = 0xFB
short s = c;   // s = 0xFFFB
```

```
char  c = 5;  // c = 0x05
short s = c;  // s = 0x0005
```


Digits (right to left)


```
void sdigits(long x,
             unsigned long b) {
  assert (b>1);
  if (x >= 0)
    digits((unsigned long)x, b);
    else {
      digits((unsigned long)(-x), b);
      printf("-");
    }
}
```


The Rationals ($\mathbb{Q}$)

There is no direct type in conventional programming languages for $\mathbb{Q}$

Whoever is happy with the rationals can jump to Notation

$\mathbb{Q}$ extra

Of course, the following is formalizing the obvious

Definition ($\sim$)

The relation $\sim$ over $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is defined by
$$\langle m_0, n_0 \rangle \sim \langle m_1, n_1 \rangle \iff m_0 \times n_1 = n_0 \times m_1$$

Definition

$$\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) \,/ \sim$$

Definition

$$\frac{m}{n} = [\langle m, n \rangle]$$

$\mathbb{Q}$ Operations extra

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Definition

$$\frac{m_0}{n_0} + \frac{m_1}{n_1} = \left[\frac{m_0 \times n_1 + m_1 \times n_0}{n_0 \times n_1}\right]$$

Definition

$$\frac{m_0}{n_0} \times \frac{m_1}{n_1} = \left[\frac{m_0 \times m_1}{n_0 \times n_1}\right]$$

Definition (Embedding $\mathbb{Z} \to \mathbb{Q}$)

$$n \mapsto \frac{n}{1}$$

Lots of things to prove above!

Wish we had Division with Remainder extra

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Lemma

Assume $0 \le x < 1$, $b \in \mathbb{N}$ and $b > 1$. Then there is unique $d \in \mathbb{N}$ and $f < 1$, where $d < b$, such that
$$x \times b = d + f.$$

Corollary

Assume $0 \le x < 1$, $b \in \mathbb{N}$ and $b > 1$. Then there is a sequence $\{d_n\}_{n=-1}^{-\infty}$ such that
$$x = \sum_{n=-1}^{-\infty} d_n \times b^n.$$

$0.\{d_n\}_{n=-1}^{-\infty}$ is the b-radix notation of x

It is not **really** infinite extra

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Definition

A notation $0.(d_n)_{n=-1}^{-\infty}$ is said to be repetitive if there is $k, l \in \mathbb{N}$ such that for each $i \in \mathbb{N}$ and $j < l$, $d_{k+i \times l+j} = d_{k+j}$

Lemma

The b-radix notation of $x \in \mathbb{Q}$, $0 \le x < 1$ is repetitive.

Proof.

For each d_n there is a corresponding f_n. For each f_n there is a corresponding r_n such that $f_n = \frac{r_n}{m}$. Thus after at most m steps there is k, n such that $r_k = r_n$, hence $f_k = f_n$ and we find the repetition. □

Notation

$$0.d_1 \cdots d_k \overline{d_{k+1} \cdots d_n} = 0.d_1 \cdots d_k d_{k+1} \cdots d_n d_{k+1} \cdots d_n \cdots$$

Example: Fraction to Decimal

Question

$$\frac{7}{25} = (?)_{10}$$

Solution

$$\frac{7}{25} \times 10 = 2 + \frac{20}{25}$$
$$\frac{20}{25} \times 10 = 8 + 0$$

Thus $\frac{7}{25} = 0.28$.

Example: Radix-5 to Decimal (1)

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Question

$$(0.34)_5 = (?)_{10}$$

Solution 1

$$(0.34)_5 = \frac{3}{5} + \frac{4}{25} = \frac{19}{25}$$

then

$$\frac{19}{25} \times 10 = 7 + \frac{15}{25}$$
$$\frac{15}{25} \times 10 = 6 + 0$$

Thus $(0.34)_5 = 0.76$.

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Example: Radix-5 fraction to Decimal (2)

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Question

$$(0.34)_5 = (?)_{10}$$

Solution 2

$$(0.34)_5 \times (20)_5 = (12)_5 + (0.3)_5$$
$$(0.3)_5 \times (20)_5 = (11)_5 + 0$$

Thus $(0.34)_5 = 0.76$

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Example: Decimal to Radix-5

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Question

$$0.5 = (?)_5$$

Solution

$0.5 \times 5 = 2 + 0.5$ // We are in infinite loop

Thus $0.5 = (0.222...)_5 = (0.\overline{2})_5$.

Imprecision is almost inevitable

since we do not possess infinite memories

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Example: Fraction to Binary

Question

$$\frac{1}{5} = (?)_2$$

Solution

$$\frac{1}{5} \times 2 = 0 + \frac{2}{5}$$
$$\frac{2}{5} \times 2 = 0 + \frac{4}{5}$$
$$\frac{4}{5} \times 2 = 1 + \frac{3}{5}$$
$$\frac{3}{5} \times 2 = 1 + \frac{1}{5}$$ here comes the inifite loop

Thus $\frac{1}{5} = (0.\overline{0011})_2$.

Example: Fraction to Binary

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Question

$$\frac{19}{28} = (?)_2$$

Solution

$$\frac{19}{28} \times 2 = 1 + \frac{10}{28} \quad \frac{20}{28} \times 2 = 1 + \frac{12}{28}$$
$$\frac{10}{28} \times 2 = 0 + \frac{20}{28} \quad \frac{12}{28} \times 2 = 0 + \frac{24}{28} \quad \frac{19}{28} = (0.10\overline{101})_2$$
$$\frac{24}{28} \times 2 = 1 + \frac{20}{28}$$

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ℚ and the Hardware

Operations

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- The operations can be reduced to ℤ-operations
- Thus ℤ notation claims can be used.

$$(0.3)_4 + (1.25)_4 = \frac{(30)_4 + (125)_4}{(100)_4}$$

$$(0.3)_4 \times (1.25)_4 = \frac{(3)_4 \times (125)_4}{(1000)_4}$$

$\mathbb{Q}$ and the Hardware

- Computer memory is finite
- Existence of maximum and minimum, as in the integers, is obvious
- In addition $\mathbb{Q}$ might require infinite digits
- Thus in addition to overflow and underflow we have imprecise calculations

Forms of $\mathbb{Q}$ representation

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- Fixed point
  - Never really processor implemented
  - Non-processor circuits might implement it
- Binary Coded Decimal
  - Ancient hardware had it
  - Software only on modern hardware
  - Important
- Floating point
  - Hardware implemented
  - Important

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Fixed Point

Left for the future

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Binary Coded Decimal

Left for the future

(Binary) Floating-Point Notation

Decimal FP is left for the future

These are all rationals, actually

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Scientific Notation

- Sometime we need some precision in a vast range
- Numbers are represented in the form

$$m \times 10^n,$$

where $|m| < 10$.

Examples

$$3.14159$$
$$6.02 \times 10^{23}$$
$$8.407564 \times 10^{-16}$$

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IEEE 754

- First standard 1985
- The driving force was Intel
- Last standard as of yet, 2019
- Before 1985 manufacturer used propriety formats
- (Also after, e.g., IBM added IEEE 754 circa 2000)
- (`float` calculations were manufacturer dependent!)

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Binary FP format

s	exponent	mantissa

The exponent is considered to be an unsigned number of w-bits width
Let $\text{bias} = 2^{w-1} - 1$

- **Normal form** $0 < \text{exponent} < 2^w - 1$:
$$(-1)^s \times 1.\text{mantissa} \times 2^{\text{exponent}-\text{bias}}$$
- **Subnormal form** $\text{exponent} = 0$:
$$(-1)^s \times 0.\text{mantissa} \times 2^{1-\text{bias}}$$
- **Bizzaro** $\text{exponent} = 2^w - 1$: NaNs, infinities, et. al.
We ignore this case in this course

Binary FP fields width

	width of		
	s	exponent	mantissa	
binary16	1	5	10	
binary32	1	8	23	float
binary64	1	11	52	double
binary128	1	15	112	
binary256	1	18	237	

Example: Decimal to binary16

Question

Present 200 in binary16

200	0
100	0
50	0
25	1
12	0
6	0
3	1
1	1
0	

$200 = (11001000)_2$

$(11001000)_2 = (1.1001000)_2 \times 2^7$

$s = 0$

$\exp = 7 + 15 = 22 = (10110)_2$

$\text{mantissa} = (1001000000)_2$

15 14 10 9 0

0	1 0 1 1 0	1 0 0 1 0 0 0 0 0 0

$200 = (0101101001000000)_{2\,\mathsf{bin16}} = (5A40)_{16\,\mathsf{bin16}}$

Example: Decimal to binary16

Question

Present $\frac{1}{5}$ in binary16

$\frac{1}{5}$	0
$\frac{2}{5}$	0
$\frac{4}{5}$	1
$\frac{3}{5}$	1
$\frac{1}{5}$	

$\frac{1}{5} = (0.\overline{0011})_2 = (1.1\overline{0011})_2 \times 2^{-3}$

$s = 0$

$\text{exponent} = -3 + 15 = 12 = (01100)_2$

$\text{mantissa} = (1001100110)_2$

15 14 10 9 0

0	0 1 1 0 0	1 0 0 1 1 0 0 1 1 0

$\frac{1}{5} \approx (0011001001100110)_{2\,\mathsf{bin16}} = (3266)_{16\,\mathsf{bin16}}$

(Carmi) Lecture 3 reached here

Example: Decimal to binary16

Present 0.00006103515625 in `binary16`

0.00006103515625	0
0.0001220703125	0
0.000244140625	0
0.00048828125	0
0.0009765625	0
0.001953125	0
0.00390625	0
0.0078125	0
0.015625	0
0.03125	0
0.0625	0
0.125	0
0.25	0
0.5	1

$$0.00006103515625 = 1 \times 2^{-14}$$
$$s = 0$$
$$\text{exponent} = (00001)_2$$
$$\text{mantissa} = (0000000000)_2$$

15	14 … 10	9 … 0
0	0 0 0 0 1	0 0 0 0 0 0 0 0 0 0

$$0.00006103515625 = (0400)_{16\,\mathsf{bin16}}$$

Example: Decimal to binary16

Present 0.000030517578125 in `binary16`

0.000030517578125	0
0.00006103515625	0
0.0001220703125	0
0.000244140625	0
0.00048828125	0
0.0009765625	0
0.001953125	0
0.00390625	0
0.0078125	0
0.015625	0
0.03125	0
0.0625	0
0.125	0
0.25	0
0.5	1

$$0.000030517578125 = 1 \times 2^{-15}$$
$$0.000030517578125 = (0.1)_2 \times 2^{-14}$$
$$s = 0$$
$$\text{exponent} = (00000)_2$$
$$\text{mantissa} = (1000000000)_2$$

15	14 … 10	9 … 0
0	0 0 0 0 0	1 0 0 0 0 0 0 0 0 0

$$0.000030517578125 = (0200)_{16\,\mathsf{bin16}}$$

binary16 Minimals

Minimal Positive

$$2^{-24} = 5.960464477539063 \times 10^{-8} = (0001)_{16\,\mathsf{bin16}}$$

Minimal Non-Negative

$$0 = (0000)_{16\,\mathsf{bin16}}$$

Example: binary16 Addition

$$(1.1)_2 \times 2^{-1} + (1.1)_2 \times 2^{1} =$$
$$(0.011)_2 \times 2^{1} + (1.1)_2 \times 2^{1} =$$
$$((0.011)_2 + (1.1)_2) \times 2^{1} =$$
$$(1.111)_2 \times 2^{1}$$

$$(1.1)_2 \times 2^{0} + (1.1)_2 \times 2^{1} =$$
$$(0.11)_2 \times 2^{1} + (1.1)_2 \times 2^{1} =$$
$$((0.11)_2 + (1.1)_2) \times 2^{1} =$$
$$(10.01)_2 \times 2^{1} =$$
$$(1.001)_2 \times 2^{2}$$

Always increase the lower exponent!

Example: `binary16` Addition Precision

$(1.1)_2 \times 2^{-10} + (1.1)_2 \times 2^1 =$
$(0.000000000011)_2 \times 2^1 + (1.1)_2 \times 2^1 =$
$((0.000000000011)_2 + (1.1)_2) \times 2^1 =$
$(1.100000000011)_2 \times 2^1 \simeq$
$(1.1000000000)_2 \times 2^1 =$
$(1.1)_2 \times 2^1$

Example: `binary16` Multiplication

$(1.1)_2 \times 2^{-1} \times (1.1)_2 \times 2^1 =$
$((1.1)_2 \times (1.1)_2) \times (2^{-1} \times 2^1) =$
$(10.01)_2 \times 2^0 =$
$(1.001)_2 \times 2^1$

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