Boolean Algebras

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January 29, 2025

Aim

- Introducing boolean algebras
- Present examples for their usage

Boolean algebra

Definition

The structure $\langle \mathbb{B}, 0, 1, +, \cdot, ^{-} \rangle$, where

- $\mathbb{B}$ is a set,
- 0 and 1 are constants (presumbly, not equal),
- $+$ and $\cdot$ are binary operators (i.e., 2-ary functions),
- $^{-}$ is a unary operator (1-ary functions),

is a boolean algebra if the following hold for each x, y, z,

Neutrality:	$x + 0 = x$	$x \cdot 1 = x$
Comutativity:	$x + y = y + x$	$x \cdot y = y \cdot x$
Associativity:	$(x + y) + z = x + (y + z)$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Distributivity:	$x \cdot (y + z) = x \cdot y + x \cdot z$	$x + (y \cdot z) = (x + y) \cdot (x + z)$
Complement:	$x + \bar{x} = 1$	$x \cdot \bar{x} = 0$ unique?

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Lemma (Uniqueness of complement)

If $xy = 0$, $x + y = 1$, $xz = 0$ and $x + z = 1$ then $y = z$.

Proof.

$$
\begin{aligned}
y =& y \cdot 1 = \\
=& y \cdot (x + z) = \\
=& y \cdot x + y \cdot z = \\
=& 0 + y \cdot z = \\
=& x \cdot z + y \cdot z = \\
=& z \cdot (x + y) = \\
=& z \cdot 1 = \\
=& z.
\end{aligned}
$$

□

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Minimal Boolean Algebra

Lemma

Assume $\mathbb{B}$ is a boolean algebra. Then the set of constants $\{0,1\}$ is closed under the operators $+$, $\cdot$ and $^{-}$. (i.e., $\{0,1\}$ is a boolean subalgebra of $\mathbb{B}$.)

Proof.

1. $^{-}$: By the relevant neutralities of 0 and 1 we have $0+1=1$ and $0\cdot 1=0$. Thus $\bar{0}=1$ and $\bar{1}=0$.
2. $\cdot$: By the neutrality of 1 we get $1\cdot 0=0\cdot 1=0$ and $1\cdot 1=1$. For $0\cdot 0$ we work as follows. $0\cdot 0=0\cdot 0+0=0\cdot 0+(0\cdot 1)=0\cdot(0+1)=0\cdot 1=0$.
3. $+$: By the neutrality of 0 we get $1+0=0+1=1$ and $0+0=0$. For $1+1$ we work as follows. $1+1=(1+1)\cdot 1=(1+1)\cdot(1+0)=1+(1\cdot 0)=1+0=1$. □

Truth table form of the lemma

x	$\bar{x}$
0	1
1	0

x	y	$x\cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

- **If** there is a boolean algebra then
  - its 0 and 1 follow the above tables
- No boolean algebra has been spotted as of yet!

The 2-Valued boolean algebra

Let us take the truth tables above as a **definition**.

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Associativity of $\cdot$

x	y	z	$x\cdot y$	$(x\cdot y)\cdot z$	$y\cdot z$	$x\cdot(y\cdot z)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

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Associativity of $+$

x	y	z	$x+y$	$(x+y)+z$	$y+z$	$x+(y+z)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

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Distributivity of $\cdot$ over $+$

x	y	z	$y+z$	$x \cdot (y+z)$	$x \cdot y$	$x \cdot z$	$x \cdot y + x \cdot z$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Distributivity of $+$ over $\cdot$

x	y	z	$y \cdot z$	$x + (y \cdot z)$	$x + y$	$x + z$	$(x+y) \cdot (x+z)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

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2-Valued Booean Algebra

Corollary

$\langle \{0,1\}, 0, 1, +, \cdot, ^{-} \rangle$ *is a boolean algebra.*

Digital Design

Only the 2-valued boolean algebra is used

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Convention

High school conventions

1. The '$\cdot$' symbol can be dropped, i.e., $xy = x \cdot y$.
2. '$\cdot$' takes precedence over '$+$', e.g., $x + yz = x + (y \cdot z)$.

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Boolean Functions

Definition

A function $f : \mathbb{B}^n \to \mathbb{B}^m$ is called a boolean function.

Note

1. $\bar{}$ is a 1-ary function
2. Both $\cdot$ and $+$ are 2-ary functions

Exclusive-or, xor ($\oplus$)

This is a function we all know and we use infix notation for it

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Definable Functions

- There might be functions without a defining formula
- This is the situation with functions $\mathbb{R} \to \mathbb{R}$

Definition

- A boolean function f is definable if it falls into one of the following cases:
  - $f = 0$ or $f = 1$ or $f = x_i$.
  - $f = (\bar{g})$, where g is a definable
  - $f = (f_0 + f_1)$ where both f_0 and f_1 are definable
  - $f = (f_0 \cdot f_1)$ where both f_0 and f_1 are definable.
  - $f = g(h_0, \ldots, h_{n-1})$ where $g, h_0, \ldots, h_{n-1}$ are definable

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Truth tables

Definable functions have a truth table representation

			$f(x,y,z) = x + yz$	
x	y	z	$y \cdot z$	$x + y \cdot z$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

EXPONENTIAL

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Why Definable Functions?

- Definable boolean functions can be realized in hardware
- Boolean functions can realize useful operations
- We need some motiviation!

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(Carmi) Lecture 4 reached here

Motivation

- Seven segment
- mod 3
- Addition

The Seven-Segment

Seven Segment

Problem

The seven segment component has seven light segments as follows:

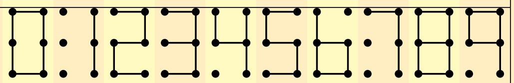
$b = c = 1, a = d = e = f = g = 0$

Devise a boolean formula accepting a decimal digit and showing it on the seven segment

The function form

- The decimal digits will be coded in binary
- Each of the segments needs its own line
- $f = \langle g, f, e, d, c, b, a \rangle : \mathbb{B}^4 \to \mathbb{B}^7$

Seven Segment (Symbols to Boolean)



	0	1	2	3	4	5	6	7	8	9
b_0	0	1	0	1	0	1	0	1	0	1
b_1	0	0	1	1	0	0	1	1	0	0
b_2	0	0	0	0	1	1	1	1	0	0
b_3	0	0	0	0	0	0	0	0	1	1
a	1	0	1	1	0	1	0	1	1	1
b	1	1	1	1	1	0	0	1	1	1
c	1	1	0	1	1	1	1	1	1	1
d	1	0	1	1	0	1	1	0	1	0
e	1	0	1	0	0	0	1	0	1	0
f	1	0	0	0	1	1	1	0	1	1
g	0	0	1	1	1	1	1	0	1	1

Seven Segment (thinking)

- There are four inputs b_3, b_2, b_1, b_0
- The target functions depend on the decimal digits
- (As is evident by the previous slide)
- Temporary functions $m_0, \ldots, m_9$ might be easier on us

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Seven Segment (temporary functions)

b_3	b_2	b_1	b_0	m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0	0	0	0	0	0
0	0	1	1	0	0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0	0	1	0	0	0	0
0	1	1	0	0	0	0	0	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	0	0	0	0	0	0	0	0	0	1

$m_0 = \bar{b}_3\bar{b}_2\bar{b}_1\bar{b}_0$ $m_3 = \bar{b}_3\bar{b}_2 b_1 b_0$ $m_6 = \bar{b}_3 b_2 b_1\bar{b}_0$ $m_9 = b_3\bar{b}_2\bar{b}_1 b_0$

$m_1 = \bar{b}_3\bar{b}_2\bar{b}_1 b_0$ $m_4 = \bar{b}_3 b_2\bar{b}_1\bar{b}_0$ $m_7 = \bar{b}_3 b_2 b_1 b_0$

$m_2 = \bar{b}_3\bar{b}_2 b_1\bar{b}_0$ $m_5 = \bar{b}_3 b_2\bar{b}_1 b_0$ $m_8 = b_3\bar{b}_2\bar{b}_1\bar{b}_0$

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Seven Segment (calculating a)

m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	a
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1

$$a = m_0 + m_2 + m_3 + m_5 + m_7 + m_8 + m_9$$
$$a = \bar{m}_1\bar{m}_4\bar{m}_6$$

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Seven Segment (calculating b)

m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	b
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1

$$b = m_0 + m_1 + m_2 + m_3 + m_4 + m_7 + m_8 + m_9$$
$$b = \bar{m}_5 \bar{m}_6$$

Seven Segment (calculating c)

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m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	c
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1

$$c = m_0 + m_1 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8 + m_9$$
$$c = \bar{m}_2$$

Seven Segment (calculating d)

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m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	d
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	0

$$d = m_0 + m_2 + m_3 + m_5 + m_6 + m_8$$
$$d = \bar{m}_1 \bar{m}_4 \bar{m}_7 \bar{m}_9$$

Seven Segment (calculating e)

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m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	d_9	e
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	0

$$e = m_0 + m_2 + m_6 + m_8$$
$$e = \bar{m}_1 \bar{m}_3 \bar{m}_4 \bar{m}_5 \bar{m}_7 \bar{m}_9$$

Seven Segment (calculating f)

m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	d_9	f
1	0	0	0	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1

$$f = m_0 + m_4 + m_5 + m_6 + m_8 + m_9$$
$$f = \bar{m}_1\bar{m}_2\bar{m}_3\bar{m}_7$$

Seven Segment (calculating g)

m_0	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	d_9	g
1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1
0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	1	1

$$g = m_2 + m_3 + m_4 + m_5 + m_6 + m_8 + m_9$$
$$g = \bar{m}_0\bar{m}_1\bar{m}_7$$

Seven Segment (notes)

- **Lots** of work
- For illegal input the output is garbage
- No mathematical laws used in the table construction
- Here, we have no way but to use truth tables
- In **this** case reading formula from the table is easy

mod 3

A mod 3 function

Problem

Devise a boolean formula for computing n mod 3 for $0 \leq n \leq 15$

The function form

- We use binary coding
- 4-bits input
- 2-bits output
- Thus the function is of the form $f = \langle f_1, f_0 \rangle : \mathbb{B}^4 \to \mathbb{B}^2$

mod 3 (truth table)

Decimal		Binary					
n	n mod 3	n_3	n_2	n_1	n_0	f_1	f_0
0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	1
2	2	0	0	1	0	1	0
3	0	0	0	1	1	0	0
4	1	0	1	0	0	0	1
5	2	0	1	0	1	1	0
6	0	0	1	1	0	0	0
7	1	0	1	1	1	0	1
8	2	1	0	0	0	1	0
9	0	1	0	0	1	0	0
10	1	1	0	1	0	0	1
11	2	1	0	1	1	1	0
12	0	1	1	0	0	0	0
13	1	1	1	0	1	0	1
14	2	1	1	1	0	1	0
15	0	1	1	1	1	0	0

How do we get formulae from this?

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Definability of Boolean Functions (SOP form) extra

Theorem

The boolean functions are definable.

Proof.

It is enough to show that the functions $f : \mathbb{B}^n \to \mathbb{B}$ are definable. We do this by induction.

$\underline{n = 0}$: A 0-ary function is a constant, that is either 0 or 1. Thus definability is immediate.

$\underline{n + 1}$: Let $f : \mathbb{B}^{n+1} \to \mathbb{B}$ be a function. Let $f_0(x_{n-1}, \ldots, x_0) = f(0, x_{n-1}, \ldots, x_0)$ and $f_1(x_{n-1}, \ldots, x_0) = f(1, x_{n-1}, \ldots, x_0)$. By induction the functions f_0 and f_1 are definable, hence the function $f(x_n, \ldots, x_0) = \bar{x}_n \cdot f_0(x_{n-1}, \ldots, x_0) + x_n \cdot f_1(x_{n-1}, \ldots, x_0)$ is definable. □

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Proof of definability (POS form) extra

Proof.

$\underline{n + 1}$: Let $f : \mathbb{B}^{n+1} \to \mathbb{B}$ be a function. Let $f_0(x_{n-1}, \ldots, x_0) = f(0, x_{n-1}, \ldots, x_0)$ and $f_1(x_{n-1}, \ldots, x_0) = f(1, x_{n-1}, \ldots, x_0)$. By induction the functions f_0 and f_1 are definable, hence the function $f(x_n, \ldots, x_0) = (x_n + f_0(x_{n-1}, \ldots, x_0)) \cdot (\bar{x}_n + f_1(x_{n-1}, \ldots, x_0))$ is definable. □

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mod 3 unwinding f_1 (step 1) extra

n_3	n_2	n_1	n_0	f_1
0	0	0	0	0
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	0
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	0
1	1	0	1	0
1	1	1	0	1
1	1	1	1	0

mod 3 unwinding f_1 (step 2) extra

$\bar{n}_3$			
n_2	n_1	n_0	f_1^0
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

n_3			
n_2	n_1	n_0	f_1^1
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

$$f_1 = \bar{n}_3 f_1^0 + n_3 f_1^1$$

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mod 3 unwinding f_1^0, f_1^1 (step 3) extra

$\bar{n}_3\bar{n}_2$		
n_1	n_0	f_1^{00}
0	0	0
0	1	0
1	0	1
1	1	0

$\bar{n}_3 n_2$		
n_1	n_0	f_1^{01}
0	0	0
0	1	1
1	0	0
1	1	0

$$f_1^0 = \bar{n}_3\bar{n}_2 f_1^{00} + \bar{n}_3\bar{n}_2 f_1^{01}$$

$n_3\bar{n}_2$		
n_1	n_0	f_1^{10}
0	0	1
0	1	0
1	0	0
1	1	1

$n_3 n_2$		
n_1	n_0	f_1^{11}
0	0	0
0	1	0
1	0	1
1	1	0

$$f_1^0 = n_3\bar{n}_2 f_1^{10} + n_3\bar{n}_2 f_1^{11}$$

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mod 3 unwinding $f_1^{00}, f_1^{01}, f_1^{10}, f_1^{11}$ (step 4) extra

$\bar{n}_3\bar{n}_2\bar{n}_1$	
n_0	f_1^{000}
0	0
1	0

$f_1^{000} = 0$

$\bar{n}_3\bar{n}_2 n_1$	
n_0	f_1^{001}
0	1
1	0

$f_1^{001} = \bar{n}_0$

$\bar{n}_3 n_2\bar{n}_1$	
n_0	f_1^{010}
0	0
1	1

$f_1^{010} = n_0$

$\bar{n}_3 n_2 n_1$	
n_0	f_1^{011}
0	0
1	0

$f_1^{011} = 0$

$n_3\bar{n}_2\bar{n}_1$	
n_0	f_1^{100}
0	1
1	0

$f_1^{100} = \bar{n}_0$

$n_3\bar{n}_2 n_1$	
n_0	f_1^{101}
0	0
1	1

$f_1^{101} = n_0$

$n_3 n_2\bar{n}_1$	
	f_1^{110}
0	0
1	0

$f_1^{110} = 0$

$n_3 n_2 n_1$	
n_0	f_1^{111}
0	1
1	0

$f_1^{111} = \bar{n}_0$

$$f_1^{00} = \bar{n}_3\bar{n}_2\bar{n}_1 f_1^{000} + \bar{n}_3\bar{n}_2 n_1 f_1^{001} \quad f_1^{01} = \bar{n}_3 n_2\bar{n}_1 f_1^{010} + \bar{n}_3 n_2 n_1 f_1^{011}$$
$$f_1^{10} = n_3\bar{n}_2\bar{n}_1 f_1^{100} + n_3\bar{n}_2 n_1 f_1^{101} \quad f_1^{11} = n_3 n_2\bar{n}_1 f_1^{110} + n_3 n_2 n_1 f_1^{111}$$

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mod 3 f_1 extra

- Note that we skipped the last unwinding
- We can do one variable, no need to go to zero variables
- But possible...

$$
\begin{aligned}
f_1 =& \bar n_3 \bar n_2 \bar n_1 f_1^{000} + \bar n_3 \bar n_2 n_1 f_1^{001} + \bar n_3 n_2 \bar n_1 f_1^{010} + \bar n_3 n_2 n_1 f_1^{011} + \\
& n_3 \bar n_2 \bar n_1 f_1^{100} + n_3 \bar n_2 n_1 f_1^{101} + n_3 n_2 \bar n_1 f_1^{110} + n_3 n_2 n_1 f_1^{111} \\
=& \bar n_3 \bar n_2 \bar n_1 \cdot 0 + \bar n_3 \bar n_2 n_1 \bar n_0 + \bar n_3 n_2 \bar n_1 n_0 + \bar n_3 n_2 n_1 \cdot 0 + \\
& n_3 \bar n_2 \bar n_1 \bar n_0 + n_3 \bar n_2 n_1 n_0 + n_3 n_2 \bar n_1 \cdot 0 + n_3 n_2 n_1 \bar n_0 \\
=& \bar n_3 \bar n_2 n_1 \bar n_0 + \bar n_3 n_2 \bar n_1 n_0 + \\
& n_3 \bar n_2 \bar n_1 \bar n_0 + n_3 \bar n_2 n_1 n_0 + n_3 n_2 n_1 \bar n_0
\end{aligned}
$$

mod 3 f_1 in canonical SOP

n_3	n_2	n_1	n_0	f_1	m_2	m_5	m_8	m_{11}	m_{14}
0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	1	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	1	0	1	1	0	1	0	0	0
0	1	1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0	0	0
1	0	0	0	1	0	0	1	0	0
1	0	0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
1	0	1	1	1	0	0	0	1	0
1	1	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0
1	1	1	0	1	0	0	0	0	1
1	1	1	1	0	0	0	0	0	0

$$
f_1(n_3,n_2,n_1,n_0) = \sum(2,5,8,11,14) = \bar n_3 \bar n_2 n_1 \bar n_0 + \bar n_3 n_2 \bar n_1 n_0 + n_3 \bar n_2 \bar n_1 \bar n_0 + n_3 \bar n_2 n_1 n_0 + n_3 n_2 n_1 \bar n_0
$$

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mod 3 f_1 in canonical POS

n_3	n_2	n_1	n_0	f_1	M_0	M_1	M_3	M_4	M_6	M_7	M_9	M_{10}	M_{12}	M_{13}	M_{15}
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
0	0	0	1	0	1	0	1	1	1	1	1	1	1	1	1
0	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	1	0	1	1	0	1	1	1	1	1	1	1	1
0	1	0	0	0	1	1	1	0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	0	1	1	1	1	1	1
0	1	1	1	0	1	1	1	1	1	0	1	1	1	1	1
1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	0	1	1	1	1	1	1	0	1	1	1	1
1	0	1	0	0	1	1	1	1	1	1	1	0	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	1	1	1	1	1	1	1	1	0	1	1
1	1	0	1	0	1	1	1	1	1	1	1	1	1	0	1
1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0

$$
\begin{aligned}
f_1(n_3,n_2,n_1,n_0) =& \prod(0,1,3,4,6,7, \\
& 9,10,12,13,15) = \\
& (n_3+n_2+n_1+n_0)\cdot \\
& (n_3+n_2+n_1+\bar n_0)\cdot \\
& (n_3+n_2+\bar n_1+\bar n_0)\cdot \\
& (n_3+\bar n_2+n_1+n_0)\cdot \\
& (n_3+\bar n_2+n_1+n_0)\cdot \\
& (n_3+\bar n_2+\bar n_1+\bar n_0)\cdot \\
& (\bar n_3+n_2+n_1+\bar n_0)\cdot \\
& (\bar n_3+n_2+\bar n_1+n_0)\cdot \\
& (\bar n_3+\bar n_2+n_1+n_0)\cdot \\
& (\bar n_3+\bar n_2+n_1+\bar n_0)\cdot \\
& (\bar n_3+\bar n_2+\bar n_1+\bar n_0)\cdot
\end{aligned}
$$

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n mod 3 formulae

SOP

$$
\begin{aligned}
f_1 =& \bar n_3 \bar n_2 n_1 \bar n_0 + \bar n_3 n_2 \bar n_1 n_0 + n_3 \bar n_2 \bar n_1 \bar n_0 + n_3 \bar n_2 n_1 n_0 + \\
& n_3 n_2 n_1 \bar n_0 \\
f_0 =& \bar n_3 \bar n_2 \bar n_1 n_0 + \bar n_3 n_2 \bar n_1 \bar n_0 + \bar n_3 n_2 n_1 n_0 + n_3 \bar n_2 n_1 \bar n_0 + \\
& n_3 n_2 \bar n_1 n_0
\end{aligned}
$$

Note

In this case the POS form is uglier

Larger n's, kind of nasty

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mod 3 (notes)

- There is an algorithm for mod
- We used it to **build** the truth table
- We did **not** implement the algorithm
- Truth table implementation is mechanical
- However, it is not practical for large n's
- Usually it is also fast (relevant when we have hardware)
- Algorithm implemetation is usually harder

Truth tables do not scale up!!!!

However, they might give us more optimized functions

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(Carmi) **Lecture 5 reached here**

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Addition

The plan:

- One bit adder (a.k.a Half Adder)
- Two bits adder
- Three bits adder?! Wrong direction
- Summing three bits (a.k.a Full Adder)
- n-Bits adder (a.k.a Binary Adder)

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One Bit Adder

Problem

Devise a formula to add two one-bit numbers

- 2-bits inputs: Maximal possible sum is 2.
- Hence $f = \langle f_1, f_0 \rangle : \mathbb{B}^2 \to \mathbb{B}^2$

		Decimal	Binary	
x	y	$x+y$	f_1	f_0
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	2	1	0

$f_1 = xy$ $\qquad$ $f_0 = x \oplus y$

Two Bits Adder

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Problem

Devise a formula to calculate the sum of two numbers each in the range 0–3

The function form

- Of course we use binary coding
- Each of the inputs is 2-bits wide
- Thus sum is 6 at most
- Thus the output is 3-bits wide
- The function is of the form $f = \langle f_2, f_1, f_0 \rangle : \mathbb{B}^4 \to \mathbb{B}^3$

2-Bits Binary Adder (truth table)

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Decimal			Binary						
x	y	$x+y$	x_1	x_0	y_1	y_0	f_2	f_1	f_0
0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	1	0	0	1
0	2	2	0	0	1	0	0	1	0
0	3	3	0	0	1	1	0	1	1
1	0	1	0	1	0	0	0	0	1
1	1	2	0	1	0	1	0	1	0
1	2	3	0	1	1	0	0	1	1
1	3	4	0	1	1	1	1	0	0
2	0	2	1	0	0	0	0	1	0
2	1	3	1	0	0	1	0	1	1
2	2	4	1	0	1	0	1	0	0
2	3	5	1	0	1	1	1	0	1
3	0	3	1	1	0	0	0	1	1
3	1	4	1	1	0	1	1	0	0
3	2	5	1	1	1	0	1	0	1
3	3	6	1	1	1	1	1	1	0

2-Bits Binary Adder (formula)

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$$\begin{aligned}
f_2 =& \bar{x}_1 x_0 y_1 y_0 + x_1 \bar{x}_0 y_1 \bar{y}_0 + x_1 \bar{x}_0 y_1 y_0 + x_1 x_0 \bar{y}_1 y_0 + \\
& x_1 x_0 y_1 \bar{y}_0 + x_1 x_0 y_1 y_0 \\
f_1 =& \bar{x}_1 \bar{x}_0 y_1 \bar{y}_0 + \bar{x}_1 \bar{x}_0 y_1 y_0 + \bar{x}_1 x_0 \bar{y}_1 y_0 + \bar{x}_1 x_0 y_1 \bar{y}_0 + \\
& x_1 \bar{x}_0 \bar{y}_1 \bar{y}_0 + x_1 \bar{x}_0 \bar{y}_1 y_0 + x_1 x_0 \bar{y}_1 \bar{y}_0 + x_1 x_0 y_1 y_0 \\
f_0 =& \bar{x}_1 \bar{x}_0 y_1 \bar{y}_0 + \bar{x}_1 \bar{x}_0 y_1 y_0 + \bar{x}_1 x_0 \bar{y}_1 \bar{y}_0 + \bar{x}_1 x_0 y_1 \bar{y}_0 + \\
& x_1 \bar{x}_0 \bar{y}_1 y_0 + x_1 \bar{x}_0 y_1 y_0 + x_1 x_0 \bar{y}_1 \bar{y}_0 + x_1 x_0 y_1 \bar{y}_0
\end{aligned}$$

The notes of mod 3 hold also here

More than Two Bits Adder

- Three bits: $f : \mathbb{B}^6 \to \mathbb{B}^4$
- The general case looks hopeless
- However, we have seen how to add numbers:
  - Long addition of representations
- Until now our method was as follows:
  - Use an algoritm to generate truth table
  - Generate a formula from the truth table

  The exponential explosion requires something else
- The formula we generate will implement the algorithm

Recalling Long Addition

c_3	c_2	c_1	c_0	0
	x_3	x_2	x_1	x_0
	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

	0		c_0		c_1		c_2
	x_0		x_1		x_2		x_3
	y_0		y_1		y_2		y_3
c_0	z_0	c_1	z_1	c_2	z_2	c_3	z_3

- Unsigned overflow: $c_3 = 1$
- Signed overflow: $c_3 \neq c_2$

Corollary

There is a function f such that for each n,

$$\langle c_n, z_n \rangle = f(c_{n-1}, x_n, y_n),$$

where $c_{-1} = 0$

Summing Three Bits (full adder)

Problem

Devise a formula to sum three bits

The form of the function

- Three bits input
- The sum is at most three
- Thus the output is two bits
- $f = \langle f_1, f_0 \rangle : \mathbb{B}^3 \to \mathbb{B}^2$

Addition of Three Bits (formula)

			Decimal	Binary	
x	y	z	$x+y+z$	f_1	f_0
0	0	0	0	0	0
0	0	1	1	0	1
0	1	0	1	0	1
0	1	1	2	1	0
1	0	0	1	0	1
1	0	1	2	1	0
1	1	0	2	1	0
1	1	1	3	1	1

- $f_0 = x \oplus y \oplus z$
- $f_1(x, y, z) = \sum(3, 5, 6, 7) = \prod(0, 1, 2, 4)$

Simplifying the SOP version

$$
\begin{aligned}
f_1 =& \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz = \\
=& (\bar{x} + x)yz + (\bar{y} + y)xz + xy(\bar{z} + z) = \\
=& yz + xz + xy
\end{aligned}
$$

Binary Adder **without** Truth Tables

Well, almost, we use the full adder

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n-Bits Binary Adder

Problem

For each n devise a function to compute the sum of two numbers each in the range $0 - 2^n - 1$

Function Form

- Of course, binary
- Each of the input numbers is n-bits wide
- The output is $n+1$-bits wide
- The form is $f = \langle f_n, \cdots, f_0 \rangle : \mathbb{B}^{2n} \to \mathbb{B}^{n+1}$

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n-Bits Binary Adder (Algorithm)

Definitions

- Let $x_{n-1} \cdots x_0$ and $y_{n-1} \cdots y_0$ be the binary representation of the two input numbers
- Let $z_n \cdots z_0$ be the binary representation of the sum
- Let $f = \langle f_1, f_0 \rangle : \mathbb{B}^3 \to \mathbb{B}^2$ be the full adder

Long addition

$$
\begin{aligned}
\langle c_0, z_0 \rangle =& \langle f_1(x_0, y_0, 0), f_0(x_0, y_0, 0) \rangle \\
\langle c_1, z_1 \rangle =& \langle f_1(x_1, y_1, c_0), f_0(x_1, y_1, c_0) \rangle \\
\vdots =& \vdots \\
\langle c_{n-1}, z_{n-1} \rangle =& \langle f_1(x_{n-1}, y_{n-1}, c_{n-2}), f_0(x_{n-1}, y_{n-1}, c_{n-2}) \rangle \\
z_n =& c_{n-1}
\end{aligned}
$$

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Adder, more formal extra

Define the functions g_0 and g_1

$$g_0(k, x_k, y_k) = \begin{cases} f_0(x_k, y_k, 0) & k = 0, \\ f_0(x_k, y_k, g_1(k-1, x_{k-1}, y_{k-1})) & 0 < k < n, \\ f_0(0, 0, g_1(n-1, x_{n-1}, y_{n-1})) & k = n \end{cases}$$

$$g_1(k, x_k, y_k) = \begin{cases} f_1(x_k, y_k, 0) & k = 0, \\ f_1(x_k, y_k, g_1(k-1, x_{k-1}, y_{k-1})) & 0 < k < n \end{cases}$$

The following holds

For each $i \leq n$, $z_i = g_0(i, x_i, y_i)$

g_1 is going to take long to compute

Extras

In the following we deal with arbitrary boolean algebras

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Field of Sets extra

Definition

The structure $\langle P, \emptyset, X, \cup, \cap, \setminus \rangle$, where $P \subseteq \mathcal{P}(X)$, is a field of sets if the following hold for each $x, y \in P$: $\emptyset \in P$, $x \cup y \in P$, $x \cap y \in P$, and $X \setminus x \in P$. (By $\setminus$ we mean the operation $X \setminus x$.)

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extra I

Lemma

The structure $\langle P, \emptyset, X, \cup, \cap, \setminus \rangle$, where $P \subseteq \mathcal{P}(X)$ is a field of sets, is a boolean algebra.

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extra II

Proof.

There is really nothing to prove here assuming one understands the meaning of the set operations $\cup$, $\cap$ and $\setminus$.

1. $x \cup \emptyset = x$ and $x \cap X = x$.
2. $x \cup y = y \cup x$ and $x \cap y = y \cap x$.
3. $(x \cup y) \cup z = x \cup (y \cup z)$ and $(x \cap y) \cap z = x \cap (y \cap z)$.
4. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ and $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.
5. $x \cup (X \setminus x) = X$ and $x \cap (A \setminus x) = \emptyset$.

□

extra

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Corollary

For each set X, the structure $\langle \mathcal{P}(X), \emptyset, X, \cup, \cap, \setminus \rangle$ is a boolean algebra.

Taking X to be the empty set in the above corollary we have $\mathcal{P}(X) = \{\emptyset, \{\emptyset\}\}$! Thus we got the 2-valued boolean algebra!! This might lead us to suspect (correctly!) that somehow every boolean algebra can be realized as a field of sets with the usual set operations!

extra

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Theorem (Stone representation theorem [?])

Each boolean algebra is isomorphic to a field of sets.

Definition

Let $\mathbb{B}$ be a boolean algebra.

1. We will say that $x \leq y$ if $x = xy$.
2. A subset $U \subsetneq \mathbb{B}$ is called an ultrafilter if the following hold:
   - 2.1 $0 \notin U$ and $1 \in U$.
   - 2.2 If $x \in U$ and $x \leq y$ then $y \in U$.
   - 2.3 If $x, y \in U$ then $x \cdot y \in U$.
   - 2.4 If $x \in B$ then either $x \in U$ or $\bar{x} \in U$.

extra

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Proof.

Let $\mathbb{B}$ be a boolean algebra. Let $X = \{U \mid U$ is an ultrafilter over $\mathbb{B}\}$. For each $b \in \mathbb{B}$ let $\mathcal{U}_b = \{U \in X \mid b \in U\}$. Let $P = \{\mathcal{U}_b \mid b \in \mathbb{B}\}$. Then $P \subseteq \mathcal{P}(X)$ and $\langle P, \emptyset, X, \cup, \cap, \setminus \rangle$ is a field of sets.
Define the function $\pi : \mathbb{B} \to P$ by letting $\pi(b) = \mathcal{U}_b$ for each $b \in \mathbb{B}$. It is not hard to check that

$$\pi : \langle \mathbb{B}, 0, 1, +, \cdot, ^{-} \rangle \to \langle P, \emptyset, X, \cup, \cap, \setminus \rangle$$

is an isomorphism. □